Two-Stage Contests with Private Information*

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"Appear weak when you are strong, and strong when you are weak" -Sun Tzu

Abstract

In perfectly discriminating contests with private information, low ability contestants prefer to appear strong while high ability contestants prefer to appear weak. In a two-stage contest, this leads to a unique symmetric equilibrium with partial pooling in the first stage. A higher output in the first contest leads to a weakly higher belief about the contestant's ability entering the second contest. We characterize this unique equilibrium when cost of effort is linear and show how the prize allocation and type distribution impact contestants' expected output, payoffs, and the probability of surprise victories.

Keywords: Private information, Dynamic contest, Contest design, Signaling

1 Introduction

In many contest environments, the contestants have private information about their productive ability or value of winning. Additionally, it is common for contestants to compete against each other repeatedly or over several stages of a dynamic contest. Examples include firms engaged in R and D contests, rent-seeking competitions between lobbyists, labor market competitions between employees within a firm, countries participating in international arms races and athletes facing each other in sporting competitions. However, the impact of private information on the behavior of contestants in these dynamic competitions is not well understood.

In this paper, we study two-stage contests in a framework designed to capture both moral hazard (hidden effort choice) and adverse selection (privately known and persistent abilities). That is, when contestants' abilities are private information, the contestants must consider the signaling effect that exerting effort in the first stage will have

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in the second stage.[1] Contrary to the conventional wisdom that all contestants want to appear strong to their opponents, high ability contestants instead prefer to appear weak. The desire to both win the current contest and be in a preferred position for a future contest creates countervailing incentives leading to partial pooling strategies where contestants' outputs are partially informative about their ability.

We focus on the simplest setting that captures the signaling incentives of multi-stage contests: two contestants, who have either low or high ability, competing in two successive perfectly discriminating all-pay contests. In each contest, the contestants exert effort with the goal of producing the most output. The player who does so wins a prize. Individual ability is privately known by each contestant, is a complement to effort and persists across the two contests. After the first contest, the output of each contestant is publicly observed and players can update their beliefs about their opponent's ability. Given this additional information, contestants choose a new level of effort for the second contest. The contestants choose their effort levels to maximize their total payoffs over the two contests.

We show there is a unique symmetric equilibrium for the two-stage contest game. The equilibrium strategies of both high and low ability contestants reflect the trade-off between success in the first contest and optimal positioning for success in the second contest. The complementarity of ability and effort would lead to a high level of output from high ability contestants and a low level of output from low ability contestants if there was only a single contest. However, entering the second contest, a contestant with high ability will always prefer his opponent to believe he is of low ability. Likewise, a contestant with low ability wants to appear to be of high ability. Concern about the outcomes in both contests leads to an equilibrium that has partial pooling in the first contest, i.e., there is a range of outputs which can be produced by either low ability or high ability contestants. Low ability contestants who produce output in this range are *bluffing* while high ability contestants who do so are *sandbagging*.[2]

Both tactics are used in the first contest with the purpose of depressing the effort of opponents in the second contest. In particular, contestants care primarily about the effort choice of opponents who have similar ability. For example, a low ability opponent will be discouraged when they observe high output in the first contest, leading them to put less effort into the second contest. Therefore, low ability contestants have an

[1] This framework captures a two-stage all-pay auction where bids are observed but values are not. The percentage of a contestant's value that is bid is hidden from other contestants because the value of winning is private information.

[2] The terms *sandbag* and *bluff* are used in the literature to describe a player signaling to his opponent that he is weak when he is actually strong and strong when he is actually weak, respectively. These terms originate from the game of poker. In poker, *sandbagging* is when a player calls or does not increase the pot when he believes he has the better hand. *Bluffing* is when a player bids up the pot when he does not think he has the best hand.

incentive to put a high level of effort into the first contest. On the other hand, a high ability opponent would increase their effort in the second contest if they observe a high output in the first contest. Therefore, high ability contestants can avoid a highly competitive second contest by exerting relatively little effort in the first contest.

We fully characterize the unique equilibrium of the two-stage contest when costs of effort are linear. We use this characterization to consider the welfare impacts of the prize allocation over the two contests and the type distributions of the contestants.

The prizes of the two contests impact the relative strength of the countervailing incentives. A large prize in the first contest reduces pooling in this contest by increasing the incentive for high ability contestants to separate from low ability contestants. This decreases the probability that a low ability contestant beats a high ability contestant in the first stage, but increases this probability in the second contest. On the other hand, a large prize in the second contest increases the incentive to pool in the first contest as positioning for the second contest becomes more important. This increases the probability of a victory by a low ability contestant in the first contest and decreases this probability in the second contest.

For a fixed prize pool, when contestants are at least as likely to have high ability as low ability, expected output is maximized and the expected payoffs of the contestants are minimized when the total prize pool is allocated to one stage of the contest. Placing the full prize pool on one contest leads to fully separating equilibrium strategies in that contest, and no effort is exerted in the other contest. Having positive prizes for both contests leads to the partial pooling equilibrium in the first contest. While bluffing in the first contest increases expected output of low ability contestants, sandbagging by high ability contestants has the opposite effect. When contestants are at least as likely to be high ability as low ability, the expected reduction in effort from high ability contestants will outweigh the increased expected effort of low ability contestants. Additionally, partial separation in the first contest will on average lead to a less competitive second contest, lowering expected output in this contest. However, when contestants are more likely to be low ability than high ability, increased output from bluffing can outweigh the impacts of sandbagging and reduced competition in the second contest.

When the prize on each contest is fixed, a larger ability ratio between high and low ability contestants decreases the pooling in the first contest. This increases the expected payoffs of high ability contestants and the likelihood high ability contestants win the first contest. On the other hand, increasing the likelihood that contestants are high ability reduces the payoffs of high ability contestants. The expected payoffs of low ability contestants are zero regardless of the type distribution or prize allocation.

Lastly, we show that the general characteristics of the unique equilibrium of the two-stage contest extends to the case where the cost of effort is convex in each contest. In this case, spreading a prize pool over the two contests can induce sufficient additional effort to outweigh the negative impacts on expected output from sandbagging in the first contest and asymmetry in the second contest.

Uniqueness of equilibrium in dynamic games with signaling is not common and stems from the countervailing incentives in the first contest. To derive this equilibrium, we use the construction in Siegel (2014) to find the unique equilibrium of the second contest sub-game for any set of abilities and beliefs that emerge from the first contest. The incentives to sandbag and bluff clearly emerge from the equilibrium payoffs in the second contest. For high ability contestants, expected payoffs strictly decrease in the other contestant's belief about their ability. For low ability contestants, these payoffs strictly increase in the opponent's belief. The desire of each type of contestant to appear as the other type not only leads to a partial pooling equilibrium in the first contest, it also rules out any additional equilibria. In signaling games, undesirable off-equilibrium path beliefs can often be used to construct additional perfect Bayesian equilibria. In this game, however, there are no beliefs that are undesirable to both high and low ability contestants. Therefore, any belief associated with the off-path output of the first contest will cause a deviation that unravels the potential equilibrium.

This paper contributes to the literature on information manipulation in dynamic competitions. Players choose optimal strategies given their realized type and the equilibrium belief function that maps actions to a distribution over types. This literature builds off of cheap talk models, a la Crawford and Sobel (1982), where players send state-dependent messages to manipulate the actions of the receiver of the message. In competitions, players often have an incentive to bluff in order to discourage opponents. This behavior is observed in labor competitions as over working before a midterm evaluation (Ederer (2010)), in dynamic auctions as jump bidding (Avery (1998)) and in Cournot competition as excess production when firm costs are uncertain (Mirman et al. (1993) and Bonatti et al. (2017)).[3] Sandbagging, as described in Rosen (1986), is often used to lull opponents into a false sense of security. This type of signal-jamming also appears as bid-shading in repeated first price auctions, see Ortega Reichert (2000) and Bergemann and Hörner (2018).

In our setting, the incentives for strong competitors to sandbag and weak competitors to bluff is due to (i) the existence of private information, (ii) the partial revelation of

[3] In complete information Tullock contests with sequential moves, increased effort in early stages reduces effort of opposing contestants that follow, see Hinnosaar (2021).

this information through actions during the competition and (iii) the all-pay nature of the contest. This two-directional strategic information manipulation is also identified in Hörner and Sahuguet (2007) where bidders are able to signal their value through a fixed jump bid prior to an all-pay auction. Bidders with moderate values will sometimes use this jump bid while bidders with high values may not. Similarly, in a multi-stage all-pay auction with elimination, Zhang and Wang (2009) shows non-existence of a separating equilibrium when winners' bids are revealed prior to the following stage. Denter et al. (2021) shows that contestants want to reveal their strength using a costly signal only if they are certain to be considerably stronger than their opponent.[4] The current paper clarifies how the incentives that arise in these settings impact the strategies of contestants within the competition itself, both before and after information is revealed.[5]

Relatedly, the information design approach, see Kamenica and Gentzkow (2011), studies optimal information strategies assuming the information designer is able to pre-commitment prior to learning their type. The literature on information design in contests has mainly focused on the optimal information disclosure policy of the contest designer. This information has including private information about contestants' types prior to a contest, as in Zhang and Zhou (2016), Chen et al. (2017), Zheng et al. (2018), Lu et al. (2018) and Serena (2021), and early-stage outcome information released during a multistage contest, as in Zhang and Wang (2009), Ederer (2010), Klein and Schmutzler (2017), Aoyagi (2010) and Breig and Kubitz (2021). Additional work has considered the optimal information disclosure policies from the perspective of the contestants. Kovenock et al. (2015) shows that independently sharing verifiable information is strictly dominated prior to a perfectly discriminating all-pay contest.[6] On the other hand, in a Tullock contest, contestants prefer to commit to sharing information when the ratio between high and low types are sufficiently close to one, see Wu and Zheng (2017).

Information about types or outcomes in contests often creates asymmetries that reduce subsequent effort. The reduction of output from asymmetric contestant types is well known, see for example Baye et al. (1993), Che and Gale (2003), Terwiesch and

[4]Heijnen and Schoonbeek (2017) shows that the incentive to share information depends on how the costly signal impacts the payoffs of the winning contestant. When pre-contest signaling has no explicit cost Kovenock et al. (2015) shows that there is always a perfect Bayesian equilibrium with no information revelation. When contestants know that they will compete in two contests against one player of high ability and one of low ability, a partial pooling equilibrium can emerge where low ability contestants do not reveal their type and high ability contestants only reveal their type with probability less than one, see Ford et al. (2020). Signaling through coalition formation prior to a competition is studied in Konrad and Morath (2018).

[5]Repeated contests with private information are also studied in Münster (2009), where contestants may or may not value winning the contest. This leads to an equilibrium where contestants who do value the contest will sometimes not participate in the first contest but the incentive to bluff is not captured.

[6]Chen (2021) allows for any information signal through an industry wide agreement and shows that for sufficiently homogeneous players, signals that induce asymmetric contests are optimal for the contestants.

Xu (2008) and Siegel (2010). It differs from the discouragement effect caused by falling behind in early stages of a contest of complete information and uncertain outcomes as in Harris and Vickers (1987) and Konrad and Kovenock (2009).[7] A contest designer can alleviate the reduction in effort from either type of discouragement by committing to bias later stages of a dynamic contest as in Ridlon and Shin (2013) and Barbieri and Serena (2018). In the current paper, the relative size of the prizes in each contest impacts the average asymmetry in the second contest but does not favor one contestant over the other.

The rest of the paper is organized as follows. Section 2 presents the two-stage contest model with linear cost of effort. In Sections 3 and 4 we characterize the equilibrium of this model by backwards induction, focusing on the second contest in Section 3 and the first contest in Section 4. In Section 5 we discuss welfare and design implications. Section 6 considers extensions of the model and Section 7 concludes. The full characterization of the equilibrium with linear cost of effort appears in Appendix A while proofs of all results in Sections 3-5 are in Appendix B.[8]

2 Model

Two ex-ante identical contestants are independently endowed with ability, a_i, for $i = 1, 2$. The probability that each contestant is high ability, $a_i = a^h$, is $\hat{\mu}$ and low ability, $a_i = a^\ell$, is $1 - \hat{\mu}$, where $\hat{\mu} \in (0, 1)$. Ability is normalized so that $a^\ell = 1$ and $a^h > 1$, and the endowment of ability is private information for each contestant. After the initial draw of types, the abilities of the contestants are fully persistent.

There are two sequential contests, denoted by $t = 1, 2$, in which the two contestants compete to produce the most output by choosing effort, $e_{it} \geq 0$. While the cost of effort is the same for high and low ability contestants and is equal to the effort exerted, effort and ability are complimentary. Contestant i's output is deterministic and defined by $x_{it} \equiv a_i e_{it}$. The contestant that produces the most output in contest t receives prize $p_t \geq 0$. If the two contestants produce the same output, then the prize is given randomly, each contestant winning with equal probability. Given the deterministic nature of output, both strategies and payoffs will be written in terms of output rather than effort in order to ease exposition.

Prior to the second contest, the outputs of each contestant from the first contest, and therefore the winner of the first contest, become public information. Contestants

[7] See Konrad (2012) for a survey of discouragement in dynamic contests under complete information.
[8] Proofs for results in Section 6 can be found in the online appendix.

use this information to update their beliefs about their opponent's ability. Denote the public belief about contestant i's ability prior to the second contest by $\mu_i(x_{i1}) \equiv \Pr(a_i = a^h | x_{i1})$. Given that outputs from the first contest are commonly observed, contestants' first order beliefs are sufficient for characterizing beliefs in the second contest.

To unify notation between periods, let η_t denote the history of the game. Then $\eta_1 = \{\emptyset\}$ and $\eta_2 = (x_{11}, x_{21})$. A strategy for contestant i is a mapping from each history to an output distribution for each ability type. Let $H_{it}(x|\eta_t)$ denote the probability that contestant i produces at most x in contest t given history η_t when the contestant is of high ability. Similarly, let $L_{it}(x|\eta_t)$ denote this output distribution when the contestant is of low ability. A strategy for player i is $\sigma_i = \{H_{it}(x|\eta_t), L_{it}(x|\eta_t)\}_{t=1,2}$. We denote the expected output distribution of contestant i given public information as $F_{it}(x|\eta_t)$.[9] Take $h_{it}(x|\eta_t)$, $\ell_{it}(x|\eta_t)$ and $f_{it}(x|\eta_t)$ to be the densities that are induced from the distribution functions $H_{it}(x|\eta_t)$, $L_{it}(x|\eta_t)$ and $F_{it}(x|\eta_t)$ respectively.[10] We denote the support of the densities of high and low ability contestants by $X^h_{it}(\eta_t) = \{x : h_{it}(x|\eta_t) > 0\}$ and $X^\ell_{it}(\eta_t) = \{x : \ell_{it}(x|\eta_t) > 0\}$.

Contestant i's expected payoff in contest t for a given strategy of player $-i$, σ_{-i}, is

$$\mathbb{E}[\pi_{it}(x_{it})|a_i, \sigma_{-i}, \eta_t] = p_t \mathbb{E}[w_i(x_{it}, x_{-it})|\sigma_{-i}, \eta_t] - \frac{x_{it}}{a_i}, \tag{1}$$

$$\text{where} \quad w_i(x_{it}, x_{-it}) = \begin{cases} 1, & x_{it} > x_{-it} \\ 0, & x_{it} < x_{-it} \\ 1/2, & x_{it} = x_{-it} \end{cases}.$$

In the two-stage game, contestant i maximizes the expected payoffs in the two contests without discounting. For a given σ_{-i}, these payoffs are

$$\mathbb{E}[\pi_i(x_{i1}, x_{i2})|a_i, \sigma_{-i}] = \mathbb{E}[\pi_{i1}(x_{i1})|a_i, \sigma_{-i}] + \mathbb{E}\left[\mathbb{E}[\pi_{i2}(x_{i2})|a_i, \sigma_{-i}, \eta_2]|x_{i1}\right]. \tag{2}$$

Best response sets for ability types $\theta = h, \ell$ are denoted by $BR^\theta_{it}(\sigma_{-i}, \eta_t)$ where

$$\begin{aligned} BR^\theta_{i2}(\sigma_{-i}, \eta_2) &= \arg\max_{x_{i2}} \mathbb{E}[\pi_{i2}(x_{i2})|a^\theta, \sigma_{-i}, \eta_2], \\ BR^\theta_{i1}(\sigma_{-i}) &= \arg\max_{x_{i1}} \mathbb{E}[\pi_i(x_{i1}, \hat{x}_{i2}(x_{i1}, x_{-i1}))|a^\theta, \sigma_{-i}], \end{aligned}$$

and $\hat{x}_{i2}(x_{i1}, x_{-i1}) \in BR^\theta_{i2}(\sigma_{-i}, (x_{i1}, x_{-i1}))$.

We restrict attention to equilibria that are symmetric. A set of strategies (σ_i, σ_{-i}) and belief functions $(\mu_i(x), \mu_{-i}(x))$ form a symmetric perfect Bayesian equilibrium (SPBE)

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[9] For example, in the first contest this expected output distribution is $F_{i1}(x|\eta_1) = \hat{\mu} H_{i1}(x|\eta_1) + (1 - \hat{\mu}) L_{i1}(x|\eta_1)$.

[10] The extended definition of density using Dirac-delta functions is invoked where necessary.

for the two-stage contest if

1. output distributions are symmetric for each ability type and history, and belief functions are symmetric: $H_{it}(x|\eta_t) = H_{-it}(x|\eta_t)$, $L_{it}(x|\eta_t) = L_{-it}(x|\eta_t)$, and $\mu_i(x) = \mu_{-i}(x)$,

2. players update beliefs according to Bayes' rule when feasible:[11]

$$\mu_i(x) = \frac{\hat{\mu} h_{i1}(x)}{\hat{\mu} h_{i1}(x) + (1-\hat{\mu})\ell_{i1}(x)}, \tag{3}$$

3. and for any history, $X^\theta_{it}(\eta_t) \subseteq BR^\theta_{it}(\sigma_{-i}, \eta_t)$.

We will characterize the equilibrium of this two-stage contest by backwards induction. In the second contest we refer to the two contestants as the strong contestant and the weak contestant, denoted $i = s, w$, where $\mu_s(x_{s1}) \geq \mu_w(x_{w2})$. This does not rule out the possibility of the weak contestant having high ability, the strong contestant having low ability, or both.

In the first contest, we denote the symmetric equilibrium output distributions as $H^*_1(x)$ and $L^*_1(x)$ where the equilibrium ex-ante expected output distribution of each contestant is $F^*_1(x) = \hat{\mu} H^*_1(x) + (1-\hat{\mu}) L^*_1(x)$. Similarly, we denote the equilibrium belief function as $\mu^*(x)$ and equilibrium strategies as $\sigma^* = \{H^*_1(x), L^*_1(x), H^*_{i2}(x|\eta_2), L^*_{i2}(x|\eta_2)\}$.

3 Second contest

Given the cost of effort is linear, a simple transformation turns the two-stage contest into a two-stage all-pay auction with independent private values that are privately known and bids that are publicly observed. Multiplying (1) through by a_i gives payoffs in each stage of an all pay-auction with bids, x_{it}, and values, $V_{it} \equiv a_i p_t$:

$$\mathbb{E}[\tilde{\pi}_{it}(x_{it})|a_i, \sigma_{-i}, \eta_t] = V_{it}\mathbb{E}[w_i(x_{it}, x_{-it})|\sigma_{-i}, \eta_t] - x_{it}. \tag{4}$$

For any history, η_2, and belief function, $\mu_i(x_{i1})$, the second contest subgame is a special case of the asymmetric all-pay auction analyzed by Siegel (2014) where the probability of contestant i having high value is equal to $\mu_i(x_{i1})$. Because values, V_{i2}, increase in type, a_i, this game has a unique equilibrium (Siegel (2014), Proposition 1).

[11] Using the extended definition of density allows agents to update their beliefs even when they see their opponent produce an output where the distribution has a mass point. For example, if $H_{i1}(x)$ has a mass point at $\tilde{x}$, while $L_{i1}(x)$ does not, this definition implies $\mu_{i2}(\tilde{x}) = 1$.

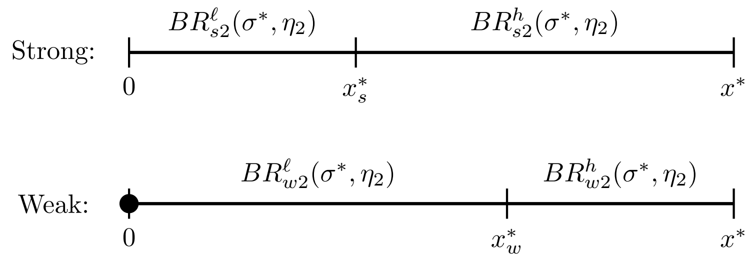

Figure 1: Representation of best response sets of the strong and weak players in the second contest.

The properties of the equilibrium of the second contest (see Siegel (2014), Lemma 1 and 2) are given below.

In the unique equilibrium, the expected output distributions of each contestant are continuous for positive outputs, and there are no outputs chosen with positive probability, except at zero. The best response sets are disjoint intervals for each ability level of a given contestant, with the set for high ability ranging over larger outputs than the set for low ability. The combined best response sets of the strong contestant and the weak contestant must be the same interval. Since the strong contestant is more likely to be of high ability, the high ability best response set is longer for the strong player. We define $x^*_i(\eta_2) \equiv \sup BR^{\ell}_{i2}(\sigma^*, \eta_2) = \inf BR^{h}_{i2}(\sigma^*, \eta_2)$ and $x^*(\eta_2) \equiv \sup BR^{h}_{s2}(\sigma^*, \eta_2) = \sup BR^{h}_{w2}(\sigma^*, \eta_2)$. We suppress the η_2 notation unless it is needed to avoid confusion. The basic structure of these best response sets is shown in Figure 1.

Equilibrium output distributions can be constructed by using indifference conditions to characterize the output densities over each region of the best response sets.[12] Given beliefs $\mu_s(x_{s1})$ and $\mu_w(x_{w1})$ these output distributions are

$$[L^*_{s2}, H^*_{s2}, L^*_{w2}, H^*_{w2}](x) = \begin{cases} \left[\frac{x}{p_2(1-\mu_s)}, 0, \frac{x}{p_2(1-\mu_w)} + \frac{a^h-1}{a^h}\frac{\mu_s-\mu_w}{1-\mu_w}, 0\right], & x \in [0, x^*_s] \\ \left[1, \frac{x-x^*_s}{p_2\mu_s}, 1 - \frac{x^*_w-x}{a^h p_2(1-\mu_w)}, 0\right], & x \in [x^*_s, x^*_w] \\ \left[1, 1 - \frac{x^*-x}{a^h p_2 \mu_w}, 1, 1 - \frac{x^*-x}{a^h p_2 \mu_w}\right], & x \in [x^*_w, x^*] \end{cases} \quad (5)$$

where $x^*_s = p_2(1-\mu_s)$, $x^*_w = p_2(1-\mu_w)$, and $x^* = p_2(\mu_w(a^h-1)+1)$.

[12] Details of this construction are given in Appendix A.2. We establish uniqueness of equilibrium in the second stage contest for an arbitrary weakly convex cost function in Section 6. Proof and characterization of the equilibrium can be found in the online appendix.

3.1 Payoffs

For strategies in the first contest, the main objects of interest from the equilibrium of the second contest are the payoffs of the contestants. A contestant's continuation value for the second contest depends on the contestant's ability as well as the public beliefs about both contestants' abilities. These payoffs are denoted as $v_i^\theta(\mu_i, \mu_{-i}) \equiv \mathbb{E}[\pi_{i2}(\hat{x}_{i2})|a^\theta, \sigma_{-i}^*, \eta_2]$, where $\mu_i = \mu^*(x_{i1})$, $\hat{x}_{i2} \in BR_{i2}^\theta(\sigma_{-i}^*, \eta_2)$ and σ_{-i}^* maps to the unique equilibrium output distribution given η_2 and a_{-i}.

Lemma 1. *Given history $\eta_2 = (x_{s1}, x_{w1})$ with associated beliefs, $\mu_s \geq \mu_w$, the second-stage contest continuation value of each contestant conditional on their ability are*

$$v_s^h(\mu_s, \mu_w) = p_2 \left(\frac{a^h - 1}{a^h}\right)(1 - \mu_w), \quad v_w^h(\mu_w, \mu_s) = p_2 \left(\frac{a^h - 1}{a^h}\right)(1 - \mu_w),$$
$$v_s^\ell(\mu_s, \mu_w) = p_2 \left(\frac{a^h - 1}{a^h}\right)(\mu_s - \mu_w), \quad v_w^\ell(\mu_w, \mu_s) = 0.$$

For contestants who are of high ability, the expected payoff is determined by the probability that the weaker contestant has high ability. Intuitively, high ability contestants are confident they can win, but increased competition, i.e., higher μ_w, increases how much effort they need to exert to do so. For contestants with low ability, expected payoffs are determined by how often the other contestant chooses no effort. While the stronger contestant will always exert positive effort, the weaker contestant exerts no effort with a probability that increases in the difference in strength between the two contestants.

Contestants can affect their perceived strength in this second contest through their choice of output in the first contest. High ability contestants prefer to look weak entering the second contest as their expected payoffs decrease when the contest appears more competitive. On the other hand, the expected payoffs of low ability contestants increase when they appear strong to their opponent. These incentives, formalized in Proposition 1, are a significant strategic force in the first contest.

Proposition 1. *Let $F_{\mu_{-i}}(M) = \Pr(\mu_{-i} \leq M)$ be the belief distribution of contestant $-i$'s ability resulting from the first contest, and let $\underline{M} = \sup\{M|F_{\mu_{-i}}(M) = 0\}$ and $\overline{M} = \inf\{M|F_{\mu_{-i}}(M) = 1\}$. For all $\mu_i \in (\underline{M}, \overline{M})$, expected payoffs in the second contest decrease for high ability players as μ_i increases, $\frac{\partial}{\partial \mu_i}\mathbb{E}[v_i^h(\mu_i, \mu_{-i})] < 0$, and increase with μ_i for low ability players, $\frac{\partial}{\partial \mu_i}\mathbb{E}[v_i^\ell(\mu_i, \mu_{-i})] > 0$.*

The marginal effect of beliefs on expected payoffs in the second contest is given by[13]

$$\frac{\partial}{\partial \mu_i}\mathbb{E}[v_i^h(\mu_i, \mu_{-i})] = -p_2 \left(\frac{a^h - 1}{a^h}\right)(1 - F_{\mu_{-i}}(\mu_i)), \tag{6}$$

$$\frac{\partial}{\partial \mu_i}\mathbb{E}[v_i^\ell(\mu_i, \mu_{-i})] = p_2 \left(\frac{a^h - 1}{a^h}\right) F_{\mu_{-i}}(\mu_i). \tag{7}$$

The distribution function of beliefs depends on contestants' conjectures about their opponent's distribution function of output in the first contest. For any set of conjectures, the incentives for high ability contestants to appear weaker are strict whenever there is a positive probability they are the weak contestant in the second contest. Likewise for low ability contestants, incentives are strict whenever they have a chance to be the strong contestant the second contest.

4 First-stage contest

In the first contest, contestants are ex-ante symmetric with probability of being high ability equal to $\hat{\mu} \in (0, 1)$. They become privately informed of their own ability prior to choosing output in the first contest. Contestants are forward looking, understanding that their output choice in the first contest will be revealed to the other contestant and subsequently will affect that contestant's beliefs and output distribution in the second contest. Given a strategy, σ_{-i}, the set of outputs that maximize the two-stage payoffs of contestant i with ability a^θ is

$$BR_{i1}^\theta(\sigma_{-i}) = \arg\max_{x_{i1}} p_1 \mathbb{E}[w_i(x_{i1}, x_{-i1})|\sigma_{-i}] - \frac{x_{i1}}{a^\theta} + \mathbb{E}[v_i^\theta(\mu_i, \mu_{-i})|x_{i1}]. \tag{8}$$

The best response sets clarify the incentives facing contestants in the first contest. High ability contestants face lower marginal cost of output and would choose higher output than low ability contestants when only considering payoffs from the first contest. However, high ability contestants prefer to look weak entering the second contest. Low ability contestants face a higher marginal cost of output but prefer to appear strong to their opponent in the second contest. When higher output in the first contest sends the signal that the contestant is more likely to be of high ability, both types of contestants face countervailing incentives. Lemma 2 establishes this relationship between first period output and beliefs in second contest.

Lemma 2. *In every SPBE, $\mu^*(x)$ is weakly increasing in x for all $x \in X_{i1} \equiv X_{i1}^h \cup X_{i1}^\ell$.*

[13] These expressions are derived in the proof of Proposition 1 in Appendix B.

If the belief function was decreasing, such that there were two outputs, $x > x'$ and $\mu^*(x) < \mu^*(x')$, then the incentives of the first and second contest would align. In this case, either high ability contestants would strictly prefer x over x', leaving x' outside this type's best response set, or low ability contestants would strictly prefer x' over x, leaving x' outside their best response set. Either situation cannot hold in equilibrium as the belief function would not satisfy Bayes' rule.

Two corner cases highlight the countervailing incentives. If there is no prize in the second stage contest, so that $p_1 > 0$ and $p_2 = 0$, then second stage continuation values are equal to zero for any history. Optimal output distributions for each contestant in the first contest are then equal to equilibrium output distributions of a hypothetical second stage contest with prize p_1 and beliefs $\mu_s = \mu_w = \hat{\mu}$ as described in (5). In this case, first period output distributions are fully separating with the best response set of the high ability contestants ranging over higher outputs than the best response set of low ability contestants.

If there is no prize in the first contest, so that $p_1 = 0$ and $p_2 > 0$, then any output distribution in the first contest with positive density on any positive output cannot be part of an equilibrium. In the unique symmetric equilibrium, the output distributions in the first contest are fully pooling, with both high and low ability contestants producing zero output with probability one. The second contest then proceeds with $\mu_w = \mu_s = \hat{\mu}$.

Proposition 2. *Given $p_1 = 0$ and $p_2 > 0$, there is a unique SPBE where $X_{i1} = \{0\}$.*

The remainder of the section characterizes the first contest output distributions of the unique symmetric equilibrium when prizes are positive in both stages.[14] In this case, the countervailing incentives of the two-stage contest lead to output distributions in the first contest that are partially pooling, and realized outputs are partially informative of contestants' abilities.

Opposing incentives for high ability contestants to appear weak and low ability contestants to appear strong in the second contest constrain the equilibrium belief function for outputs both on and off the equilibrium path of play. Often in games where players can signal private information, there are equilibria where actions off the equilibrium path are not taken because players who do so are assumed to have negative characteristics.[15] However, there is no universally negative belief that can be assigned to an off-path output. Because the belief function must be well behaved for all potential outputs, the first period equilibrium strategies conform to nice properties

[14] In the characterization of the equilibrium with positive prizes in Appendix A, it is shown that as p_1 converges to zero, the first-stage equilibrium converges to the first-stage equilibrium in Proposition 2.

[15] See, for example, Spence (1973).

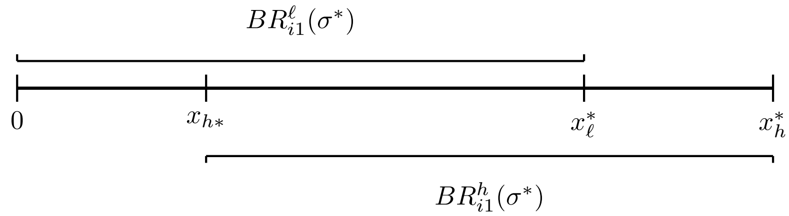

Figure 2: Representation of best response sets of the high ability and low ability contestants in the first contest.

without additional refinements. In particular, first contest output densities have no atoms (Lemma 3) and best response sets are intervals (Lemma 4).

Lemma 3. *Let $p_1 > 0$. For any SPBE, first contest output distributions are continuous and therefore $\mathbb{E}[w_i(x_{i1}, x_{-i1})|x_{i1}] = F^*_1(x_{i1}) = \hat{\mu} H^*_1(x_{i1}) + (1 - \hat{\mu}) L^*_1(x_{i1})$ is continuous.*

From Lemma 3, in any SPBE the best response sets of the high and low ability contestants can be written in terms of the expected output distribution, $F^*_1(x_{i1})$, and are given by

$$BR^h_{i1}(\sigma^*) = \{x \mid p_1 F^*_1(x) + \mathbb{E}[v^h_i(\mu_i(x), \mu_{-i})] - \frac{x}{a^h} = K^h(p_1, p_2)\} \text{ and} \tag{9}$$

$$BR^\ell_{i1}(\sigma^*) = \{x \mid p_1 F^*_1(x) + \mathbb{E}[v^\ell_i(\mu_i(x), \mu_{-i})] - x = K^\ell(p_1, p_2)\}, \tag{10}$$

where $K^h(p_1, p_2)$ and $K^\ell(p_1, p_2)$ are the expected payoffs of high and low ability contestants in the two-stage contest. These sets partition outputs of the first contest into at most three distinct intervals (see Figure 2).

Lemma 4. *Let $p_1, p_2 > 0$. Define $x_{\ell *} = \inf X^\ell_{i1}$, $x^*_\ell = \sup X^\ell_{i1}$, $x_{h*} = \inf X^h_{i1}$, and $x^*_h = \sup X^h_{i1}$. In any SPBE, the best response sets of low and high ability contestants in the first contest are intervals with $BR^\ell_{i1}(\sigma^*) = [0, x^*_\ell]$, $BR^h_{i1}(\sigma^*) = [x_{h*}, x^*_h]$ and $0 = x_{\ell *} \leq x_{h*} < x^*_\ell \leq x^*_h$.*

Incentives from the second contest guarantee that the intersection of the two best response sets is not degenerate. Both types of contestants may benefit from mimicking the other type by choosing an output in the other type's best response set. If the intersection was degenerate, this deviation can have an arbitrarily small impact on first contest payoffs while providing a discrete improvement in the continuation value due to the jump in beliefs. Similar logic requires the equilibrium belief function $\mu^*(x)$ to be continuous on the union of the best response sets, $[0, x^*_h]$.

Lemma 5. *In any SPBE, the belief function $\mu^*(x)$ is continuous in output on $[0, x_h^*]$, is weakly increasing on (x_{h*}, x_ℓ^*), takes a value of zero for all $x \in [0, x_{h*}]$ when $x_{h*} > 0$, and takes a value of one for all $x \in [x_\ell^*, x_h^*]$ when $x_h^* > x_\ell^*$.*

The above lemmas allow complete characterization of the output distributions on $[0, x_h^*]$. The following result shows that this characterization is unique and implies the uniqueness of the equilibrium belief function.

Theorem 1. *Let $p_1, p_2 > 0$. There is a unique SPBE of the two stage all pay auction where*

$$F_1^*(x) = \begin{cases} \frac{x}{p_1} & 0 \le x < x_{h*} \\ \frac{a^h}{a^h-1} - \left(\frac{a^h}{a^h-1} - \frac{x_{h*}}{p_1}\right) e^{-\frac{a^h-1}{a^h p_1}(x - x_{h*})} & x_{h*} \le x \le x_\ell^* \\ \frac{1}{p_1}\left(\frac{x}{a^h} + K^h(p_1,p_2) - \mathbb{E}[v_s^h(1, \mu_{-i})]\right) & x_\ell^* < x \le x_h^* \end{cases}$$

$$\textit{and} \quad \mu^*(x) = \begin{cases} 0 & 0 \le x < x_{h*} \\ \frac{x - x_{h*}}{p_2} + \mu^*(x_{h*}) & x_{h*} \le x \le x_\ell^* \\ 1 & x_\ell^* < x \le x_h^* \end{cases}.$$

The expected output distribution and belief function can be characterized in each of the three intervals. Output distributions for high and low ability contestants follow directly from $H_1^*(x) = \int_0^x \frac{\mu^*(t)}{\hat{\mu}} f_1^*(t)dt$ and $L_1^*(x) = \int_0^x \frac{1-\mu^*(t)}{1-\hat{\mu}} f_1^*(t)dt$.

For outputs in the range of $0 \le x < x_{h*}$, $\mu^*(x) = 0$ and therefore $\mathbb{E}[v_i^\ell(0, \mu_{-i})] = 0$. From (10), $0 \in BR_{i1}^\ell(\sigma^*)$ implies $K^\ell(p_1, p_2) = 0$ and $F_1^*(x) = \frac{x}{p_1}$. Similarly, for outputs in the range $x_\ell^* < x \le x_h^*$, $\mu^*(x) = 1$ and using (9) the distribution function is $F_1^*(x) = \frac{1}{p_1}(x/a_h + K^h(p_1,p_2) - \mathbb{E}[v_s^h(1, \mu_{-i})])$, where $\mathbb{E}[v_s^h(1, \mu_{-i})]$ is the expected payoff in the second contest for a contestant who is known to be of high ability.

All outputs in the range $x_{h*} \le x \le x_\ell^*$ are in the best response set of both low and high ability contestants and therefore satisfy both (9) and (10). Further, both types of contestants are indifferent to increasing their output over this interval leading to the conditions

$$\frac{\partial BR_{i1}^h(\sigma^*)}{\partial x} : p_1 f_1^*(x) + \frac{d\mu^*(x)}{dx}\frac{(a^h-1)p_2}{a^h}(F_\mu^*(\mu^*(x)) - 1) = \frac{1}{a^h} \quad \text{and} \tag{11}$$

$$\frac{\partial BR_{i1}^\ell(\sigma^*)}{\partial x} : p_1 f_1^*(x) + \frac{d\mu^*(x)}{dx}\frac{(a^h-1)p_2}{a^h} F_\mu^*(\mu^*(x)) = 1, \tag{12}$$

where $F_\mu^*(\cdot)$ is the symmetric equilibrium belief distribution that results from the first contest. Taking the difference of (12) and (11), it follows that the belief function must satisfy (13) over the intersection of best response functions.

$$\frac{d\mu^*(x)}{dx}\frac{\left(a^h-1\right)p_2}{a^h}=\frac{a^h-1}{a^h} \tag{13}$$

Because the marginal cost of increasing output for the low ability contestant is always more than for the high ability contestant, the marginal benefit must also be higher for the low ability contestant. Since the benefit of increasing output is the same for each type in the first contest, this difference must come from the second contest continuation values. Specifically, as shown in Equation (13), the difference in marginal benefit of appearing stronger in the second contest from choosing a higher output in the first contest must equal the difference in the marginal cost of output in the first contest.

Therefore, the belief function is strictly increasing over this interval and $F_\mu^*(\mu^*(x)) = F_1^*(x)$. Combining (13) with (12) results in the differential equation that the expected output function must satisfy over this interval.

$$p_1 f_1^*(x) = 1 - \frac{a^h-1}{a^h}F_1^*(x) \tag{14}$$

The family of solutions is $F_1^*(x) = Be^{-\frac{x(a^h-1)}{a^h p_1}} + \frac{a^h}{a^h-1}$ with boundary condition $F_1^*(x_{h*}) = \frac{x_{h*}}{p_1}$. It follows that $B = \left(\frac{x_{h*}}{p_1} - \frac{a^h}{a^h-1}\right)e^{\frac{x_{h*}(a^h-1)}{a^h p_1}}$. For a given value of x_{h*}, the unique solution to this differential equation is

$$F_1^*(x) = \frac{a^h}{a^h-1} - \left(\frac{a^h}{a^h-1} - \frac{x_{h*}}{p_1}\right)e^{-\frac{a^h-1}{a^h p_1}(x-x_{h*})}.$$

The proof of Theorem 1 requires showing that only one such x_{h*} is possible in equilibrium. Full characterization of the equilibrium requires pinning down the endpoints of the best response sets. How these endpoints are determined depends on the distribution of abilities and size of the prizes in each stage. Details of this characterization for any set of parameters is given in Appendix A. Output distributions for $a^h = 2$, $\hat{\mu} = 1/2$, and $p_1 = p_2 = 1$ are depicted in Figure 3.

5 Welfare

Characterization of the unique symmetric equilibrium allows us to consider the impacts of prize and type distributions on welfare relevant outcomes. To calculate the expected payoffs of contestants, we note that the output x_{h*} is in the best response set of both

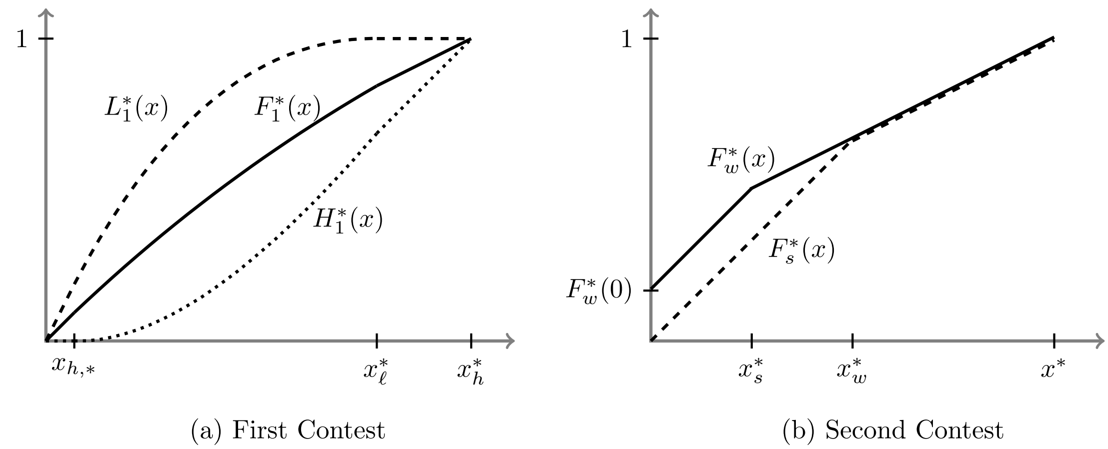

Figure 3: (a) High ability, low ability and expected output distributions in the first contest; (b) Expected output distributions of the strong and weak contestants in the second contest. Parameters taken to be $a^h = 2$, $\hat{\mu} = 1/2$, and $p_1 = p_2 = 1$.

types of contestants for any prize and type distribution. For each ability type, ex-ante expected payoffs over the two-stage game can be computed using expected payoffs of choosing x_{h*} in the first stage.

Lemma 6. *The expected payoff in the two-stage contest for a low ability contestant is zero for any prize allocation over two contests. The expected payoff of a high ability contestant is*

$$K^h(p_1, p_2) = \frac{(a^h - 1)\left(x_{h*} + p_2\left(1 - \mu^*(x_{h*})\right)\right)}{a^h}.$$

The expected payoffs of a contestant is equal to their expected prize winnings from the two stages less their expected effort. Then the ex-ante expected output of a contestant can be expressed in terms of the winning probabilities and expected payoffs of a high ability contestant. We denote the equilibrium probability that a contestant with ability a^θ wins the t-th stage of the contest as $W_t^\theta(p_1, p_2)$.

Lemma 7. *Let the expected revenue of a high ability player be* $R^h(p_1, p_2) \equiv p_1 W_1^h(p_1, p_2) + p_2 W_2^h(p_1, p_2)$. *Then the expected output of each contestant is*

$$Y(p_1, p_2) = \frac{p_1 + p_2}{2} + (a^h - 1)\hat{\mu} R^h(p_1, p_2) - a^h \hat{\mu} K^h(p_1, p_2).$$

This relationship in Lemma 7 makes it clear that expected output is higher when expected payoffs of the contestants are lower and when a high ability contestant expects more prize money, all else equal. We explore the impact of the prizes in each contest on these outcomes in Section 5.1.

5.1 Prize Allocation

The prizes in each contest impact the strategies of the first contest and the information available in the second contest. A change in p_1 impacts the marginal benefit of output in the first contest while a change in p_2 impacts the continuation values for a given belief entering the second contest. Increasing the prize in the first contest increases the incentive for high ability contestants to separate from low ability contestants. As a result, public beliefs about contestants' abilities are on average more separated entering the second contest. A larger prize in the second contest increases the incentive of high ability contestants to sandbag and low ability contestants to bluff. Additional pooling in the first contest, on average, reveals less information about opponents' abilities prior to the second contest.

Lemma 8. *Let $F^*_\mu(M)$ be the equilibrium belief distribution associated with prize ratio p_1/p_2 and $\tilde{F}^*_\mu(M)$ be associated with $\tilde{p}_1/\tilde{p}_2$. Then $p_1/p_2 > \tilde{p}_1/\tilde{p}_2$ implies $F^*_\mu(M) <_{SOSD} \tilde{F}^*_\mu(M)$.*

The ratio of prizes in the two contests impact both the efficiency and output of the two contests. When the first contest has large stakes relative to the second contest, the winner of the first contest is more likely to be of high ability. As a result, the second contest will be, on average, more asymmetric in terms of beliefs about contestants' expected abilities. This increased asymmetry increases the probability that the winner of the second contest is of low ability.

Proposition 3. *For $p_1/p_2 > \tilde{p}_1/\tilde{p}_2$, $W^\ell_1(p_1, p_2) < W^\ell_1(\tilde{p}_1, \tilde{p}_2)$ and $W^\ell_2(p_1, p_2) > W^\ell_2(\tilde{p}_1, \tilde{p}_2)$.*

Combined with ex-ante symmetry, which requires $\frac{1}{2} = \hat{\mu}W^h_t(p_1, p_2)+(1-\hat{\mu})W^\ell_t(p_1, p_2)$, the above result directly implies that a higher price ratio, p_1/p_2, decreases the probability that a low ability contestant wins the first contest and increases the probability that a high ability contestant wins the first contest. On the other hand, it increases the probability that a low contestant player wins the second contest and decreases the probability a high ability contestant wins the second contest.

Additionally, a higher prize ratio increases the probability of a "surprise victory" in the first contest, i.e., a low ability contestant beating a high ability contestant, and decreases its likelihood in the second contest. This follows in the symmetric equilibrium because low ability contestants always beat other low ability contestants with one half probability in each contest, ex-ante. Surprise victories can be thought of as reducing efficiency of allocation in settings where the designer wishes the winner to be of high ability, such as a promotion contest.

When considering a fixed prize budget, placing the entire budget on one stage or the other, essentially reducing the contest to a single stage, eliminates surprise victories in the stage with a positive prize. This prize allocation maximizes the expected revenue of a high ability contestant. The following result shows that it also minimizes the expected payoffs of contestants when $\hat{\mu} \geq 1/2$ and therefore, by Lemma 7, maximizes their expected output. Figure 4 depicts the expected output of each contestant and expected payoffs of high ability contestants for different prize allocations when $\hat{\mu} = 1/2$ and $a^h = 2$.

Proposition 4. *When $\hat{\mu} \geq 1/2$ and $p_1 + p_2 = \bar{p}$, expected total output over the two contests is maximized and expected payoffs for high ability contestants are minimized when either $p_1 = \bar{p}$ and $p_2 = 0$ or $p_1 = 0$ and $p_2 = \bar{p}$.*

Total expected output over the two contests with positive prizes is impacted both by the partial pooling strategies in the first contest and the information available in the second contest. While sandbagging and bluffing have opposite impacts in the first contest, the reduction in expected effort from high ability contestants has a larger impact on output than the increase in expected effort from low ability contestants, all else equal. Moreover, partial pooling in the first contest can frequently lead to a second contest where beliefs are asymmetric and therefore competition is reduced, lowering the expected output.

Considering the payoffs of high ability contestants, the two components of the partial pooling strategy, sandbagging and bluffing, are complementary to each other. For a given output in the first stage, bluffing by low ability contestants allows a high ability contestant to look weaker when producing that output. When contestants are at least as likely to be of high ability, this complementarity results in higher payoffs for high ability contestants when there are prizes in both contests relative to placing the entire prize budget on one stage.

When there are more low ability types on average then high ability types, the effect of bluffing on expected output can dominate the other effects. In this case, spreading the prize budget between two contests can both increase expected output and reduce the expected payoffs of high ability contestants relative to placing the entire budget on one stage of the contest. For example, when $a^h = 1.25$ and $\hat{\mu} = 0.15$, then $Y(1/2, 1/2) = 0.5032$ and $K^h(1/2, 1/2) = 0.1490$ while $Y(1, 0) = 0.5028$ and $K^h(1, 0) = 0.1700$. The following result shows that for small enough $\hat{\mu}$, splitting the budget evenly between two contests increases the expected output and decreases the expected payoffs of contestants.

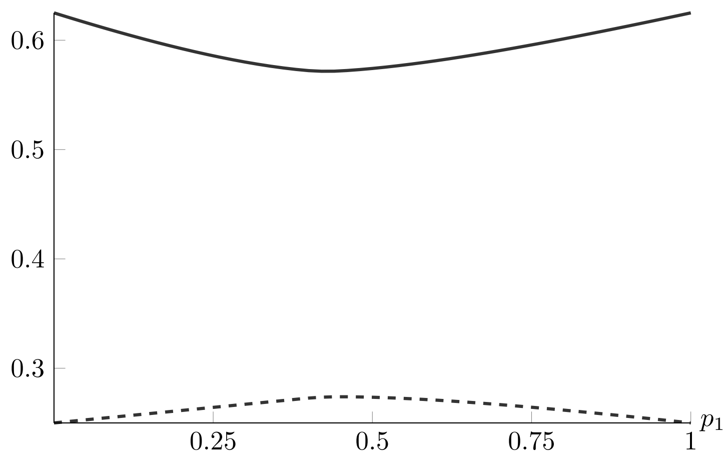

Figure 4: Expected payoffs of the high ability contestant (dashed) and ex-ante expected output of each contestant (solid) for different price ratios when $p_1 + p_2 = 1$, $\hat{\mu} = 1/2$ and $a^h = 2$.

Corollary 1. *For all $a^h > 1$ and $p_1 + p_2 = \bar{p}$, there exists $\hat{\mu}^*(a^h) > 0$ such that for all $0 < \hat{\mu} < \hat{\mu}^*(a^h)$, $Y(\bar{p}/2, \bar{p}/2) > Y(\bar{p}, 0)$ and $K^h(\bar{p}/2, \bar{p}/2) < K^h(\bar{p}, 0)$.*

In an analogous model with complete information, there is no relationship between the prize value in one contest with the strategies and information available in the other contest. Strategies in each contest would follow those in the second contest of the current model where ability types are commonly known: $\mu_i \in \{0, 1\}$ for $i = 1, 2$. In this case, expected output and expected payoffs only depend on the total prize purse, $\bar{p}$, and do not change with the relative values of p_1 and p_2 when $p_1 + p_2 = \bar{p}$.

5.2 Type Distribution

A larger ability ratio, captured by a larger a^h, reduces the amount of pooling in first stage. This change increases the probability that a high ability contestant wins the first contest, but there are competing effects in the second contest. A larger ability ratio favors high ability contestants, but a more dispersed belief function often leads to asymmetric beliefs in the second contest, improving the chances of low ability contestants, all else equal (see Proposition 3). Overall, high ability contestants enjoy higher expected payoffs when the ability ratio is higher.

Proposition 5. *Fixing p_1, p_2 and $\hat{\mu}$, let $F^*_\mu(M)$ be the equilibrium belief distribution associated with ability ratio a^h and $\tilde{F}^*_\mu(M)$ be associated with $\tilde{a}^h$. Then $a^h > \tilde{a}^h$ implies $F^*_\mu(M) <_{SOSD} \tilde{F}^*_\mu(M)$. Higher a^h also increases the expected payoffs of contestants and increases the chance that a high ability contestant wins the first contest.*

On the other hand, when contestants are more likely to be of high ability, i.e., $\hat{\mu}$ is closer to 1, high ability contestants have lower expected payoffs.

Proposition 6. *Fixing p_1, p_2 and a^h, let $F^*_\mu(M)$ be the equilibrium belief distribution associated with probability of high ability $\hat{\mu}$ and $\tilde{F}^*_\mu(M)$ be associated with probability of high ability $\tilde{\mu}$. Then $\hat{\mu} > \tilde{\mu}$ implies $\tilde{F}^*_\mu(M) <_{FOSD} F^*_\mu(M)$. Higher $\hat{\mu}$ also reduces the expected payoffs of high ability contestants.*

6 Extensions

In this section, we show the uniqueness of equilibrium result extends to perfectly discriminating all-pay contests where the cost of effort is convex. We show that the general properties of the partial pooling strategies hold in this equilibrium, and we discuss the implications on welfare and optimal contest design when the cost of effort is strictly convex. Lastly, we consider what insights from the two-stage contest model extend to a contest with more than two stages.

6.1 Generalized cost

We denote the cost function of effort as $c(e)$, which is the same for high and low ability contestants. Therefore, in terms of output, the cost function for a contestant with ability a^θ is $c(x/a^\theta)$. The cost function is assumed to be twice differentiable on the non-negative reals, strictly increasing and weakly convex, with $c(0) = 0$. Proofs of the results in this section are in the online appendix.

For any set of beliefs $\mu_w \leq \mu_s$, the second contest best response sets of the high and low ability contestants take on the same structure as in the linear cost case (see Figure 1). Specifically, the best response sets are disjoint intervals for each ability level of a given contestant, with the set for high ability contestants ranging over larger outputs than the set for low ability. The combined best response sets must be the same interval for both the strong and weak contestant.

Proposition 7. *For every history η_2, there is a unique set of output distributions $(L^*_{s2}(x|\eta_2), H^*_{s2}(x|\eta_2), L^*_{w2}(x|\eta_2), H^*_{w2}(x|\eta_2))$, that satisfy the equilibrium conditions, where $BR^\ell_{i2}(\sigma^*, \eta_2) = [0, x^*_i]$, $BR^h_{i2}(\sigma^*, \eta_2) = [x^*_i, x^*]$ for $i = s, w$ and $0 \leq x^*_s \leq x^*_w \leq x^*$.*

As in the linear cost case, the ex-ante output distribution and belief function in the first stage contest can be characterized in each of three intervals partitioned by the best response sets. Given the equilibrium output distribution and belief function, the best response sets of the high and low ability contestants are

$$BR_{i1}^{h}(\sigma^*) = \{x | p_1 F_1^*(x) + \mathbb{E}[v_i^h(\mu^*(x), \mu_{-i})] - c\left(\frac{x}{a^h}\right) = K^h(p_1, p_2)\} \text{ and}$$
$$BR_{i1}^{\ell}(\sigma^*) = \{x | p_1 F_1^*(x) + \mathbb{E}[v_i^{\ell}(\mu^*(x), \mu_{-i})] - c(x) = K^{\ell}(p_1, p_2)\}.$$

All outputs in the range $x_{h*} \leq x \leq x_{\ell}^*$ must be in the best response set of both low and high ability contestants. Subtracting the condition for $BR_{i1}^{\ell}(\sigma^*)$ from the condition for $BR_{i1}^{h}(\sigma^*)$ gives

$$\mathbb{E}[v_i^h(\mu^*(x), \mu_{-i})] - \mathbb{E}[v_i^{\ell}(\mu^*(x), \mu_{-i})] = c\left(\frac{x}{a^h}\right) - c(x) + K^h(p_1, p_2). \tag{15}$$

The marginal impact of the belief on the continuation values of contestants are

$$\frac{\partial}{\partial \mu_i} E[v_i^h(\mu_i, \mu_{-i})] = d(\mu_i)(F_{\mu_{-i}}(\mu_i) - 1) \text{ and } \frac{\partial}{\partial \mu_i} E[v_i^{\ell}(\mu_i, \mu_{-i})] = d(\mu_i) F_{\mu_{-i}}(\mu_i),$$

where $d(\mu_i) \equiv \left[p_2 + \frac{\partial}{\partial \mu_i} c\left(\frac{c^{-1}(p_2(1-\mu_i))}{a_h}\right)\right]$. Then taking the derivative of each side of (15) with respect to output gives the generalized version of equation (13),

$$\frac{d\mu^*(x)}{dx}\left[p_2 + \frac{\partial}{\partial \mu_i} c\left(\frac{c^{-1}(p_2(1-\mu_i))}{a_h}\right)\right] = c'(x) - \frac{1}{a_h} c'\left(\frac{x}{a^h}\right). \tag{16}$$

Combining this with the derivative of the best response condition of the high ability contestants gives (17), the differential equation that the output density function must satisfy over this interval which gives the generalized version of (14),

$$p_1 f_1^*(x) = c'(x)(1 - F_1^*(x)) + \frac{1}{a_h} c'\left(\frac{x}{a_h}\right) F_1^*(x). \tag{17}$$

For $x \in [0, x_{h*})$, $\mu^*(x) = 0$, $\mathbb{E}[v_i^{\ell}(\mu^*(x), \mu_{-i})] = 0$ and $p_1 F_1^*(x) = c(x)$. For all $x \in (x_{\ell}^*, x_h^*]$, $\mu^*(x) = 1$, $\mathbb{E}[v_i^h(\mu^*(x), \mu_{-i})] = \mathbb{E}[v_i^h(1, \mu_{-i})]$ and $p_1 F_1^*(x) + \mathbb{E}[v_i^h(1, \mu_{-i})] = c(x/a^h) + K^h(p_1, p_2)$.

Theorem 2. *Let $p_1, p_2 > 0$ and let $c(e)$ be twice differentiable on the non-negative reals, strictly increasing and weakly convex, with the cost of zero effort being zero. There is a unique SPBE of the two-stage contest.*

Construction of the equilibrium output distributions in each contest and the belief function of the first contest are given in the online appendix for $c(e) = ke^{\alpha}$ and $\hat{\mu} = 1/2$. The output distributions of each contest stage for quadratic cost of effort are given in Figure 5.

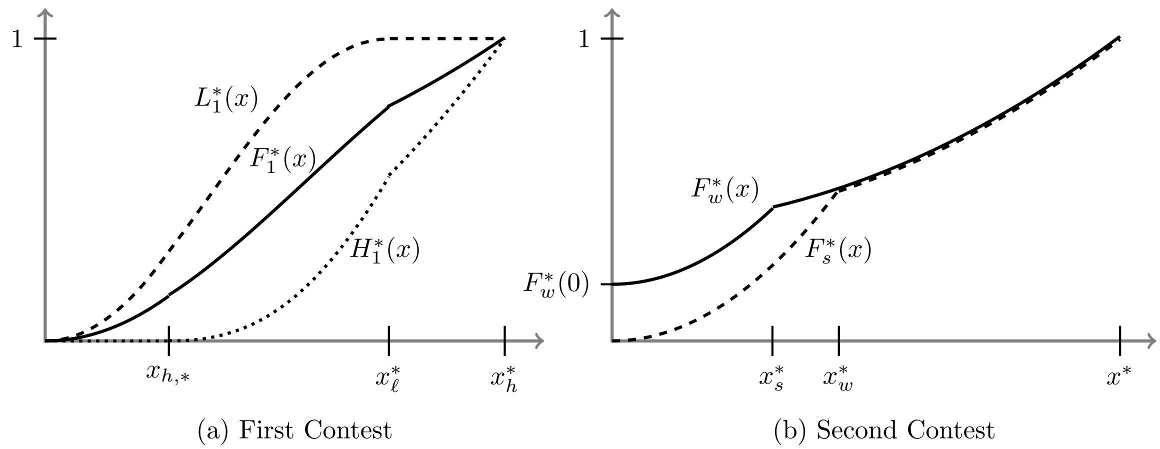

Figure 5: (a) High ability, low ability and expected output distributions in the first contest with $c(e) = e^2$; (b) Expected output distributions of the strong and weak contestants in the second contest with $c(e) = e^2$. Parameters are taken to be $a^h = 2$, $\hat{\mu} = 1/2$, and $p_1 = p_2 = 1/2$.

6.1.1 Welfare

As in the linear cost case, the expected payoff of a low ability contestant in the two-stage contest is zero. Moreover, x_{h*} is still in the best response set of the high ability player and two-stage expected payoffs are

$$\begin{aligned} K^h(p_1, p_2) &= p_1 F_1^*(x_{h*}) - c\left(\frac{x_{h*}}{a^h}\right) + \mathbb{E}[v_i^h(\mu(x_{h*}), \mu_{-i})] \\ &= c(x_{h*}) - c\left(\frac{x_{h*}}{a^h}\right) + p_2(1 - \mu(x_{h*})) - c\left(\frac{c^{-1}(p_2(1 - \mu(x_{h*})))}{a^h}\right). \end{aligned}$$

With convex cost of effort, the expected output depends not only on the expected payoffs and revenue of a high ability contestant, it also depends on how contestants allocate their effort between the two periods. High effort in one contest and low effort in the second is more costly to the contestant then a similar effort in each stage that produces the same total output.

Fixing the total prize, we numerically investigate the impact of how prizes are allocated between the first and second contests and compare this to the linear cost case. An example of expected output and expected payoffs of high ability contestants for convex cost of effort is given in Figure 6.

When the cost of effort in each contest is sufficiently convex, an extension of Proposition 4 does not hold. Expected output is increased when the prize purse is spread over two contests rather than placed entirely in one stage of the contest. At the same time, as in the linear cost case, an asymmetric allocation of prizes avoids a second contest

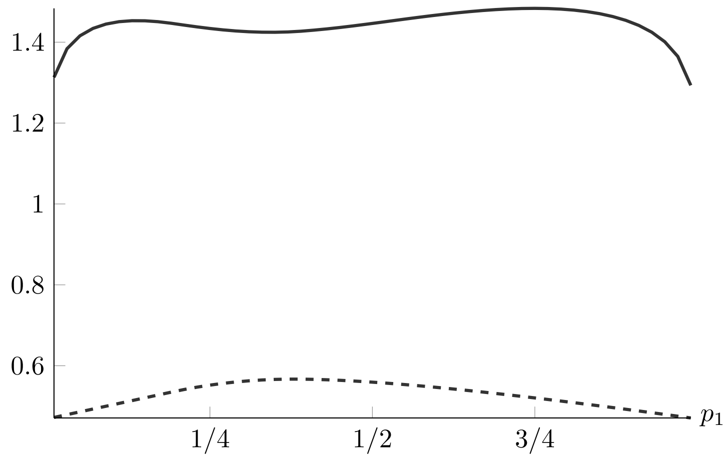

Figure 6: Expected payoffs of the high ability contestant (dashed) and ex-ante expected output (solid) of each contestant for different price ratios when $p_1 + p_2 = 1$, $\hat{\mu} = 1/2$ and $a^h = 2$, $c(e) = e^4$.

with both a large prize and low level of competition. Allocating a majority of the prize to the first contest minimizes the impact of lower competition in the second contest, while allocating a majority of the prize to the second contest reduces the likelihood of lessened competition in the second contest by increasing pooling in the first contest. Numerical simulation suggests two local maxima for these two asymmetric allocations of prizes. To maximize "excitement" contest designers can have a low stakes initial stage followed by high stakes in the second stage. If identification of a high ability contestant is more important, then a high stakes initial competition (choosing a leader) can be followed by lower stakes to continue encouraging effort (sticking by the leader almost certainly).

As in the linear cost case, payoffs for the high ability contestants are higher when there are prizes in both stages than when the full prize is in one contest due to the complimentary of the sandbagging and bluffing strategies. When cost is convex, then numerical analysis suggests that allocating a larger prize to the second contest maximizes the expected payoffs of the contestants. The intuition is as follows: as the cost becomes more convex, the difference in marginal cost for a low ability and high ability contestant for a given output increases. Then, for a given prize allocation over the two contests, there will be less pooling in the first contest. To increase pooling to the point where payoffs are maximized requires more prize to be allocated to the second contest and less to the first.[16]

[16]The extension of Lemma 8 to convex costs where $c(e) = ke^{\alpha}$ is given and proved in the online appendix.

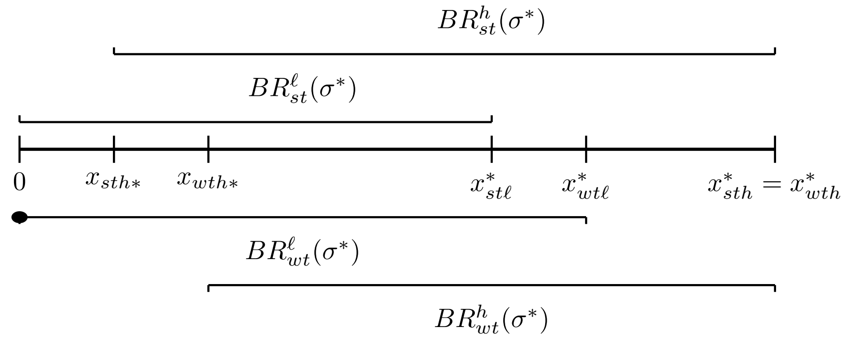

Figure 7: Representation of best response sets of the high ability and low ability contestants in stage $t = 2, \ldots, T-1$ with $0 < \mu_{wt} < \mu_{st} < 1$.

6.2 Multi-stage contests

Consider a T-stage contest where the prize in each contest is positive, $p_t > 0$ for $t = 1, \ldots, T$. For a given strategy, σ_{-i}, and history, η_t, denote the continuation payoffs in stage $t = 1, \ldots, T-1$ as $v^\theta_{it}(\mu_{it}, \mu_{-it}) \equiv \mathbb{E}[\pi_{it}(\hat{x}_{it}) + v^\theta_{i,t+1}(\mu_{i,t}(\hat{x}_{it}), \mu_{-i,t+1})|a^\theta, \sigma_{-i}, \eta_t]$, where $\mu_{it} = \mu_{i,t-1}(x_{i,t-1})$ and $\hat{x}_{it} \in BR^\theta_{it}(\sigma_{-i}, \eta_t)$. The continuation payoffs in stage T in any SPBE must be the same as the second contest payoffs in the unique SPBE of the two-stage contest described in Lemma 1.

Given history η_t, contestants in stage $t \geq 2$ will not in general have symmetric beliefs about each other's ability. Let the public belief that each contestant is high ability be given by $\mu_{st} \geq \mu_{wt}$, where the strong contestant in period t may differ from the strong contestant in other stages. Best response sets for each contestant $i = s, w$ are

$$BR^h_{it}(\sigma_{-i}, \eta_t) = \{x \mid p_t F_{-it}(x) + \mathbb{E}[v^h_{i,t+1}(\mu_{it}(x), \mu_{-i,t+1})] - c\left(\frac{x}{a^h}\right) = v^h_{it}(\mu_{it}, \mu_{-it})\} \text{ and}$$
$$BR^\ell_{it}(\sigma_{-i}, \eta_t) = \{x \mid p_t F_{-it}(x) + \mathbb{E}[v^\ell_{i,t+1}(\mu_{it}(x), \mu_{-i,t+1})] - c(x) = v^\ell_{it}(\mu_{it}, \mu_{-it})\}.$$

In what follows, we describe equilibrium play in stage t under the assumption that continuation values in stage $t+1$ have the properties in Proposition 1 which are satisfied by the second-stage continuation values in the two-stage contest. In words, we assume that high ability contestants prefer to look weak and low ability contestants prefer to look strong entering stage $t+1$. In this case, each contestant will use a partial pooling strategy in stage t.[17] The output distribution of contestant $-i$ depends on the best response sets and the value of the belief function of contestant i. For $x \in BR^\ell_{it}(\sigma_{-i}, \eta_t) \setminus BR^h_{it}(\sigma_{-i}, \eta_t)$, $\mu^*_{it}(x) = 0$ and therefore $p_t f^*_{-it}(x) = c'(x)$. For $x \in BR^h_{it}(\sigma_{-i}, \eta_t) \setminus BR^\ell_{it}(\sigma_{-i}, \eta_t)$, $\mu^*_{it}(x) = 1$, and therefore $p_t f^*_{-it}(x) = \frac{1}{a^h} c'\left(\frac{x}{a^h}\right)$. For

[17]This follows the arguments outlined in the proof of Lemma 4.

$x \in BR^h_{it}(\sigma_{-i}, \eta_t) \cup BR^\ell_{it}(\sigma_{-i}, \eta_t)$, $p_t f^*_{-it}(x) \in \left[\frac{1}{a^h} c'\left(\frac{x}{a^h}\right), c'(x)\right]$. We further assume that $f^*_{-it}(x)$ is decreasing in $\mu^*_{it}(x)$ in this interval. This is true in the first stage of the two-stage contest.[18]

Let $x_{it\theta*} = \inf BR^\theta_{it}(\sigma_{-i}, \eta_t)$ and $x^*_{it\theta} = \sup BR^\theta_{it}(\sigma_{-i}, \eta_t)$. Given $\mu_{st} > \mu_{wt}$ and $x^*_{sth} = x^*_{wth}$, it follows that $\mu_{st}(x) \geq \mu_{wt}(x)$ and $f^*_{wt}(x) \leq f^*_{st}(x)$ for all $x > 0$ where the inequalities are strict for some x. This implies that $F^*_{wt}(0) > 0$ and $F^*_{st}(0) = 0$ as well as $x_{sth*} < x_{wth*}$ and $x^*_{st\ell} < x^*_{wt\ell}$.[19] Representative best response sets are given in Figure 7.

Using the properties of the best response sets and the assumptions on the continuation payoffs in stage $t+1$, we investigate the impact of beliefs in period t on expected payoffs in period t. This exercise, similar to that in the second contest of the two-stage contest in Section 3.1, provides insights into the incentives to look strong or weak entering the t-th stage of the contest.

A higher μ_{st} for a fixed μ_{wt} implies that $\mu_{st}(x)$ must increase for some $x \in [0, x^*_{ith}]$ which further decreases $f^*_{wt}(x)$ relative to $f^*_{st}(x)$ for at least some values of $x > 0$. This requires both $F^*_{wt}(0)$ and x^*_{ith} to increase. Given $0 \in BR^\ell_{it}(\sigma_{-i}, \eta_t)$ and $x^*_{ith} \in BR^h_{it}(\sigma_{-i}, \eta_t)$, this change raises the t-stage payoffs of a low ability contestant that is strong, and it reduces the t-stage payoffs of high ability contestants that are either weak or strong. Similarly, a lower μ_{wt} for a fixed μ_{st} further increases $f^*_{st}(x)$ relative to $f^*_{wt}(x)$ for some positive outputs. This again requires $F^*_{wt}(0)$ to increase, but higher $f^*_{st}(x)$ decreases x^*_{ith}. Therefore, lower μ_{wt} increases the t-stage payoffs of high ability contestants that are either strong or weak and increases the payoffs of low ability contestants that are strong. Combining these effects, high ability contestants benefit in stage t by looking weak entering the t-th contest and low ability contestants benefit in stage t by looking strong.

7 Conclusion

In this paper, we investigate the behavior of contestants in a multi-stage contest when they have private information about their ability or value of winning. We show that low ability contestants prefer that opponents believe they are high ability while high ability contestants prefer the opposite. These incentives lead low ability contestants to

[18]See (17) and note that in the asymmetric case $F^*_{\mu_{-i},t}(\mu^*_{it}(x)) = F^*_{-it}(x)$. This property comes from the fact that a low ability contestant's incentive to look strong increases with $\mu_{it}(x)$ while a high ability contestant's incentive to look weak decreases with $\mu_{it}(x)$.

[19]These relationships follow from arguments similar to those for stage two of the two-stage contest, see the proof of Proposition 7 in the online appendix.

bluff by exerting more effort and high ability contestants to sandbag by exerting less effort in the first stage of a two-stage contest.

Sandbagging and bluffing impact the welfare of the contestants and the contest designer. While sandbagging decreases the output that comes from high ability contestants, bluffing increases output from low ability contestants. Because high ability contestants on average produce higher output, a contest designer that wishes to maximize expected output (and minimize contestant rents) should usually avoid spreading a prize pool over multiple stages. Instead, the designer should implement a single contest. However, if contestants are more likely to be low ability than high ability, the positive effect from bluffing can outweigh the negative effects from sandbagging and the potential discouragement of low ability contestants in the second stage of the contest. Moreover, when cost of effort is convex in each stage, a multi-stage contest may increase output relative to a single contest. However, simulations suggest that splitting prizes evenly lowers expected output relative to a contest with stages that have asymmetric prizes.

Appendices

A Equilibrium Characterization

A.1 First Stage Contest

From Theorem 1, the first stage expected output distribution and belief function is

$$F_1^*(x) = \begin{cases} \frac{x}{p_1}, & 0 \le x < x_{h*} \\ \frac{a^h}{a^h-1} - \left(\frac{a^h}{a^h-1} - \frac{x_{h*}}{p_1}\right) e^{-\frac{a^h-1}{a^h p_1}(x - x_{h*})}, & x_{h*} \le x \le x_\ell^* \\ \frac{1}{p_1}\left(\frac{x}{a^h} + K^h(p_1,p_2) - \mathbb{E}[v_s^h(1,\mu_{-i})]\right), & x_\ell^* < x \le x_h^* \end{cases}$$

$$\text{and } \ \mu^*(x) = \begin{cases} 0, & 0 \le x < x_{h*} \\ \frac{x - x_{h*}}{p_2} + \mu^*(x_{h*}), & x_{h*} \le x \le x_\ell^* \\ 1, & x_\ell^* < x \le x_h^* \end{cases}.$$

Given $h_1^*(x) = \frac{1}{\hat{\mu}} f_1^*(x)\mu^*(x)$ and $\ell_1^*(x) = \frac{1}{1-\hat{\mu}}(1-\mu^*(x))f_1^*(x)$, the distribution functions for high and low ability contestants are

$$H_1^*(x) = \begin{cases} 0, & 0 \le x \le x_{h*} \\ \frac{1}{\hat{\mu}}\tilde{H}_1^*(x), & x_{h*} \le x \le x_\ell^* \\ 1 - \frac{1-F_1^*(x)}{\hat{\mu}}, & x_\ell^* \le x \le x_h^* \end{cases} \qquad L_1^*(x) = \begin{cases} \frac{F_1^*(x)}{1-\hat{\mu}}, & 0 \le x \le x_{h*} \\ \frac{1}{1-\hat{\mu}}\tilde{L}_1^*(x), & x_{h*} \le x \le x_\ell^* \\ 1, & x_\ell^* \le x \le x_h^* \end{cases},$$

where

$$\tilde{H}_1^*(x) = F_1^*(x)\mu^*(x) - \frac{a^h}{a^h-1}\left(\mu^*(x) - \mu^*(x_{h*}) - \frac{p_1}{p_2}(F_1^*(x) - F_1^*(x_{h*}))\right) \quad \text{and}$$

$$\tilde{L}_1^*(x) = F_1^*(x)(1-\mu^*(x)) + \frac{a^h}{a^h-1}\left(\mu^*(x) - \mu^*(x_{h*}) - \frac{p_1}{p_2}(F_1^*(x) - F_1^*(x_{h*}))\right).$$

Characterization of the equilibrium can be broken down into cases depending on how the best response sets of the two types of contestants overlap. From Lemma 4 for any $p_1, p_2 > 0$, the intersection of the two best response sets is non-degenerate. When p_1 is large relative to p_2 then there are three distinct intervals, an interval that is only in the best response set of low ability contestants, another that is just in the best response set of high ability contestants, and the intersection of the two best response sets. The case where there are exactly two intervals can happen in two different ways depending on the values of the parameters a^h and $\hat{\mu}$. Specifically, when $\hat{\mu} > \frac{1}{\log(a^h)} - \frac{1}{a^h-1}$ the two-interval case has $0 = x_{h*} < x_\ell^* < x_h^*$. When $\hat{\mu}$ is below this threshold the two-interval case has $0 < x_{h*} < x_\ell^* = x_h^*$.

The threshold, $\overline{\mu}(a^h) = \frac{1}{\log(a^h)} - \frac{1}{a^h-1}$ is never above $\frac{1}{2}$ as $\frac{\partial \overline{\mu}}{\partial a^h} < 0$ and $\lim_{a^h \to 1^+} \overline{\mu} = \frac{1}{2}$ and $\lim_{a^h \to \infty} \overline{\mu} = 0$. The limits are calculated using L'hospitals rule, and the derivative being negative follows from the inequality $\log(1+x) \le \frac{x}{\sqrt{x+1}}$ for $x > 0$. Below, we detail the construction of the equilibrium in these two cases.

A.1.1 High likelihood of high ability contestants ($\hat{\mu} \geq \overline{\mu}(a^h)$)

When p_1 is small enough relative to p_2, then the incentives to pool are strong, so that the best response sets of the high ability and low ability contestants completely overlap. This is the case whenever

$$\frac{p_1}{p_2} \leq \frac{1-\hat{\mu}}{\frac{a^h}{a^h-1}\left(\frac{a^h}{a^h-1}\log(a^h)-1\right)} \equiv \underline{p}^h. \tag{18}$$

When the parameters satisfy this condition, $x_{h*}=0$ and $x_\ell^*=x_h^*$. Combining $F_1^*(x_h^*)=1$ and $H_1^*(x_h^*)=1$ we get the following conditions

$$\hat{\mu} = \mu^*(x_h^*) - \frac{a^h}{a^h-1}\left((\mu^*(x_h^*)-\mu^*(0)) - \frac{p_1}{p_2}\right) \quad \text{and} \tag{19}$$

$$1 = \frac{a^h}{a^h-1}\left(1-e^{-\frac{a^h-1}{a^h p_1}x_h^*}\right). \tag{20}$$

If follows from (20) that $x_h^* = p_1\frac{a^h}{a^h-1}\log(a^h)$. Using $\mu^*(x)=x/p_2+\mu^*(0)$, $\mu(x_h^*)-\mu(0) = \frac{a^h}{a^h-1}\frac{p_1}{p_2}\log(a^h)$. Plugging this into (19), beliefs at the endpoints of the best response sets are

$$\mu^*(x_h^*) = \hat{\mu} + \frac{p_1}{p_2}\frac{a^h}{a^h-1}\left(\frac{a^h}{a^h-1}\log(a^h)-1\right) \quad \text{and} \tag{21}$$

$$\mu^*(0) = \hat{\mu} - \frac{p_1}{p_2}\frac{a^h}{a^h-1}\left(1-\frac{1}{a^h-1}\log(a^h)\right). \tag{22}$$

For all a^h and $\hat{\mu}$, and for small enough p_1, $x_{\ell*}=x_{h*}=0$ and $x_\ell^*=x_h^*=p_1\frac{a^h}{a^h-1}\log(a^h)$. It follows that as $p_1 \to 0$ the supremum and infimum of the best response sets of both low and high ability contestants approach 0. Additionally from (21) and (22), $\mu^*(x_h^*) \to \hat{\mu}$ and $\mu^*(0) \to \hat{\mu}$. Therefore, when the first-stage prize converges to zero, the first-stage equilibrium converges to the equilibrium when $p_1 = 0$ described in Proposition 2; both high and low ability contestants choose zero output with probability one.

When $\frac{p_1}{p_2} > \underline{p}^h$, then $\mu^*(x_\ell^*)=1$ and $x_h^* > x_\ell^*$. Using $L(x_\ell^*)=1$ and $\mu^*(x_\ell^*)=1$,

$$1-\hat{\mu} = \frac{a^h}{a^h-1}\left(1-\mu^*(x_{h*}) - \frac{p_1}{p_2}(F_1^*(x_\ell^*)-F_1^*(x_{h*}))\right).$$

Plugging in the output distribution from Theorem 1,

$$\frac{p_2}{p_1}\frac{a^h-1}{a^h}(1-\hat{\mu}) = \frac{p_2}{p_1}(1-\mu^*(x_{h*})) - \left(\frac{a^h}{a^h-1}-\frac{x_{h*}}{p_1}\right)\left(1-e^{-\frac{a^h-1}{a^h}\frac{p_2}{p_1}(1-\mu^*(x_{h*}))}\right). \tag{23}$$

From Lemma 5, if $\mu^*(x_{h*}) > 0$ then $x_{h*} = 0$. Alternatively, if $x_{h*} > 0$, then $\mu^*(x_{h*}) = 0$. The two cases are split when both $\mu^*(x_{h*}) = 0$ and $x_{h*} = 0$ and (23) at

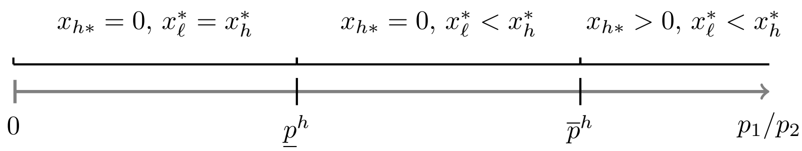

Figure 8: Representation of cases for differing prize ratios when $\hat{\mu} > \overline{\mu}(a^h)$.

this cutoff is

$$\frac{p_1}{p_2}\frac{a^h}{a^h-1}\left(1-e^{-\frac{a^h-1}{a^h}\frac{p_2}{p_1}}\right) = 1 - \frac{a^h-1}{a^h}(1-\hat{\mu}).$$

There is a unique price-ratio cutoff, call it $\overline{p}^h$, for each $\hat{\mu}$ and a^h: the left hand side is increasing from 0 to 1 as p_1/p_2 goes from 0 to infinity, and the right hand side is a constant between 0 and 1. This cutoff is decreasing in $\hat{\mu}$, which implies that a larger p_1/p_2 is required to be in the three-interval case.

For $p_1/p_2 > \overline{p}^h$, (23) becomes

$$\frac{p_2}{p_1}\frac{a^h-1}{a^h}(1-\hat{\mu}) = \frac{p_2}{p_1} - \left(\frac{a^h}{a^h-1} - \frac{x_{h*}}{p_1}\right)\left(1-e^{-\frac{a^h-1}{a^h}\frac{p_2}{p_1}}\right).$$

In this case, solving for x_{h*} and using the fact that $\mu(x_\ell^*) = 1$, the endpoints of the middle interval are

$$x_{h*} = p_1\frac{a^h}{a^h-1} - p_2\frac{1-\frac{a^h-1}{a^h}(1-\hat{\mu})}{1-e^{-\frac{p_2}{p_1}\frac{a^h-1}{a^h}}} \text{ and } x_\ell^* = p_2 + x_{h*}.$$

Note that $\frac{x_{h*}}{p_1}$ is increasing in $\frac{p_1}{p_2}$. From Lemma 6 and second-stage expected payoffs, the constant on the third interval equals

$$K^h(p_1,p_2) - \mathbb{E}[v_s^h(1,\mu_{-i})] = \frac{a^h-1}{a^h}(x_{h*} + p_2\hat{\mu}).$$

The equilibrium output distribution at x_h^* gives $K^h(p_1,p_2) - \mathbb{E}[v_s^h(1,\mu_{-i})] = p_1 - \frac{x_h^*}{a^h}$ and therefore the upper endpoint is $x_h^* = p_1 a^h - (a^h-1)(x_{h*} + p_2\hat{\mu})$.

For $\underline{p}^h < p_1/p_2 \leq \overline{p}^h$, $x_{h*} = 0$ and (23) becomes

$$(1-\mu^*(x_{h*})) - \frac{a^h-1}{a^h}(1-\hat{\mu}) = \frac{p_1}{p_2}\frac{a^h}{a^h-1}\left(1-e^{-\frac{a^h-1}{a^h}\frac{p_2}{p_1}(1-\mu^*(x_{h*}))}\right).$$

Note that as p_1/p_2 increases, $\mu^*(x_{h*})$ decreases. Because $x_{h*} = 0$, the belief function implies $x_\ell^* = p_2(1-\mu^*(x_{h*}))$, or directly

$$x_\ell^* - p_1\frac{a^h}{a^h-1}\left(1-e^{-\frac{a^h-1}{a^h p_1}x_\ell^*}\right) = \frac{p_2(1-\hat{\mu})(a^h-1)}{a^h}.$$

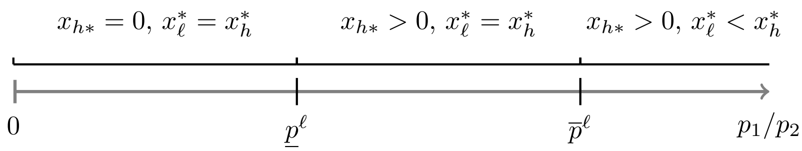

Figure 9: Representation of cases for differing prize ratios when $\hat{\mu} < \overline{\mu}(a^h)$.

This expression uniquely pins down x^*_ℓ, and therefore $\mu^*(0)$, implicitly. From Lemma 6 and second stage expected payoffs, the constant on the second interval equals $K^h(p_1, p_2) - \mathbb{E}[v^h_s(1, \mu_{-i})] = p_2 \frac{a^h-1}{a^h}(\hat{\mu} - \mu^*(0))$. Then $F(x^*_h) = 1$ gives $x^*_h = p_1 a^h - p_2(a^h - 1)(\hat{\mu} - \mu^*(0))$.

A.1.2 Low likelihood of high ability contestants ($\hat{\mu} \leq \overline{\mu}(a^h)$)

When $\hat{\mu} \leq \overline{\mu}(a^h)$ and prizes satisfy (24), then $x_{h*} = 0$ and $x^*_\ell = x^*_h$. The construction is as in the high $\hat{\mu}$ case with $x^*_h = p_1 \frac{a_h}{a_h - 1} \log(a_h)$ and beliefs at the endpoints described by (21) and (22).

$$\frac{p_1}{p_2} \leq \frac{\hat{\mu}}{\frac{a^h}{a^h-1}\left(1 - \frac{1}{a^h-1}\log(a^h)\right)} \equiv \underline{p}^\ell \tag{24}$$

When $p_1/p_2 > \underline{p}^\ell$ we have that $x_{h*} > 0$, $\mu^*(x_{h*}) = 0$. The output distribution of the high ability contestant becomes

$$H^*_1(x) = \frac{1}{\hat{\mu}}\left(F^*_1(x)\mu^*(x) - \frac{a^h}{a^h - 1}\left(\mu^*(x) - \frac{p_1}{p_2}\left(F^*_1(x) - F^*_1(x_{h*})\right)\right)\right).$$

The cut off between the two and three-interval case is when $F(x^*_\ell) = H^*_1(x^*_\ell) = 1$ and $\mu^*(x^*_\ell) = 1$. The first two conditions imply that

$$\hat{\mu} = \mu^*(x^*_\ell) - \frac{a^h}{a^h - 1}\left(\mu^*(x^*_\ell) - \frac{p_1}{p_2}\left(1 - \frac{x_{h*}}{p_1}\right)\right).$$

Combining this condition with $\mu^*(x^*_\ell) = 1$ and $F(x^*_\ell) = 1$ which is

$$1 = \frac{a^h}{a^h - 1} - \left(\frac{a^h}{a^h - 1} - \frac{x_{h*}}{p_1}\right) e^{-\frac{p_2}{p_1}\frac{a^h-1}{a^h}\mu^*(x^*_\ell)},$$

we get that the price ratio cutoff, $\overline{p}^\ell$, is determined by

$$\frac{p_1}{p_2}\left(\frac{a^h}{a^h - 1} - 1\right)\left(e^{\frac{p_2}{p_1}\frac{a^h-1}{a^h}} - 1\right) = 1 - (1 - \hat{\mu})\frac{a^h - 1}{a^h}. \tag{25}$$

The right hand side is constant with respect to the price ratio and is between 1 and $\frac{1}{a^h}$ depending on the value of $\hat{\mu}$. The left hand side of (25) strictly decreases in p_1/p_2 from higher than 1 and approaches $\frac{1}{a^h}$ as the prize ratio goes to infinity. Therefore there is a single positive value of p_1/p_2 that satisfies this equality.

When $\underline{p}^\ell < p_1/p_2 < \overline{p}^\ell$, then $\mu^*(x^*_\ell) < 1$ and is implicitly determined by

$$\frac{p_1}{p_2}\left(\frac{a^h}{a^h-1}-1\right)\left(e^{\frac{a^h-1}{a^h}\frac{p_2}{p_1}\mu^*(x^*_\ell)}-1\right) = \mu^*(x^*_\ell) - (\mu^*(x^*_\ell)-\hat{\mu})\frac{a^h-1}{a^h}. \tag{26}$$

The endpoints of the two intervals are $x_{h*} = p_1 + p_2\frac{a^h-1}{a^h}(\mu^*(x^*_\ell)-\hat{\mu}) - p_2\mu^*(x^*_\ell)$ and $x^*_\ell = p_1 + p_2\frac{a^h-1}{a^h}(\mu^*(x^*_\ell)-\hat{\mu})$.

When $p_1/p_2 > \overline{p}^\ell$, then we are in the three-interval case and the construction follows as it did in the high $\hat{\mu}$ case with identical characterization of the endpoints of the intervals.

A.2 Second Stage Contest

Characterization of the second stage for any set of beliefs that follows the first stage are given in Section 3. The output distributions are derived from indifference conditions on each contestant's best response set for each type. For example, the strong contestant with high ability must be indifferent to picking all outputs between x^*_s and x^*. For all $x, x' \in [x^*_s, x^*]$,

$$p_2F^*_{w2}(x) - \frac{x}{a^h} = p_2F^*_{w2}(x') - \frac{x'}{a^h}.$$

Rearranging and taking the limit as $x \to x'$, $\lim_{x\to x'}\frac{F^*_{w2}(x)-F^*_{w2}(x')}{\frac{x}{a^h}-\frac{x'}{a^h}} = \frac{1}{p_2}$. Then the output density of the weak contestant on this interval is $f^*_{w2}(x') = \frac{1}{p_2a^h}$. Similar conditions on each interval for each contestant allows us to characterize the densities of the output their entire support.

- $x^*_w \le x \le x^*$: $h^*_{s2}(x) = \frac{1}{p_2a^h\mu_s}$, $h^*_{w2}(x) = \frac{1}{p_2a^h\mu_w}$, $f^*_{s2}(x) = f^*_{w2}(x) = \frac{1}{p_2a^h}$.
- $x^*_s \le x \le x^*_w$: $h^*_{s2}(x) = \frac{1}{p_2\mu_s}$, $\ell^*_{w2}(x) = \frac{1}{p_2a^h(1-\mu_w)}$, $f^*_{s2}(x) = \frac{1}{p_2}$, $f^*_{w2}(x) = \frac{1}{p_2a^h}$.
- $0 \le x \le x^*_s$: $\ell^*_{s2}(x) = \frac{1}{p_2(1-\mu_s)}$, $\ell^*_{w2}(x) = \frac{1}{p_2(1-\mu_w)}$, $f^*_{s2}(x) = f^*_{w2}(x) = \frac{1}{p_2}$.

It remains to characterize the cutoff points, x^*_w, x^*_s and x^*, and $L_w(0)$. In equilibrium, the distributions of output for each contestant must satisfy

$$L^*_{i2}(x^*_i) = 1, \quad H^*_{i2}(x^*_i) = 0, \quad F^*_{i2}(x^*_i) = 1-\mu_i, \text{ and } \quad F^*_{i2}(x^*) = 1.$$

The strong contestant chooses no effort with zero probability, so using $L^*_{s2}(x^*_s) = 1$, $L^*_{s2}(0) = 0$ and the definition of $\ell^*_{s2}(x)$ on $[0, x^*_s]$,

$$\int_0^{x^*_s}\ell^*_{s2}(x)dx = L^*_{s2}(x^*_s) - L^*_{s2}(0) = \frac{x^*_s}{p_2(1-\mu_s)} = 1.$$

Then $x^*_s = p_2(1-\mu_s)$. Similarly, $x^*_w = p_2(1-\mu_w)$. From these endpoints we can calculate x^* using

$$\int_{x_s^*}^{x_w^*} h_{s2}^*(x)dx = \frac{x_w^* - x_s^*}{\mu_s} = \frac{(1-\mu_w)-(1-\mu_s)}{\mu_s} = \frac{\mu_s - \mu_w}{\mu_s},$$

$$\int_{x_w^*}^{x^*} h_{s2}^*(x_s)dx = 1 - \frac{\mu_s - \mu_w}{\mu_s} = \frac{\mu_w}{\mu_s}, \quad \text{and}$$

$$\int_{x_w^*}^{x^*} f_{s2}^*(x_s)dx = \frac{1}{p_2}\left(\frac{x^*}{a^h} - \frac{p_2(1-\mu_w)}{a^h}\right) = \mu_w,$$

$$\Rightarrow \quad x^* = p_2(\mu_w(a^h - 1) + 1).$$

Lastly, we pin down the probability that the weaker contestant exerts no effort.

$$\int_{x_s^*}^{x_w^*} \ell_{w2}^*(x)dx = \frac{p_2(1-\mu_w) - p_2(1-\mu_s)}{a^h p_2(1-\mu_w)} = \frac{\mu_s - \mu_w}{a^h(1-\mu_w)} \quad \text{and}$$

$$\int_0^{x_s^*} \ell_{w2}^*(x)dx = \frac{p_2(1-\mu_s)}{p_2(1-\mu_w)} = \frac{1-\mu_s}{1-\mu_w}$$

$$\Rightarrow \quad L_{w2}^*(0) = 1 - \frac{1-\mu_s}{1-\mu_w} - \frac{\mu_s - \mu_w}{a^h(1-\mu_w)} = \frac{a^h - 1}{a^h}\frac{\mu_s - \mu_w}{1-\mu_w}.$$

B Proofs

Proof of Lemma 1

The expected payoffs of a high ability contestant are equal to the value of winning less the cost of producing output x^*, as producing x^* guarantees a win:

$$v_s^h(\mu_s, \mu_w) = v_w^h(\mu_w, \mu_s) = p_2 \left(1 - \frac{\mu_w(a^h - 1) + 1}{a^h}\right) = p_2 \left(\frac{a^h - 1}{a^h}\right)(1 - \mu_w).$$

The expected payoffs of low ability contestants is equal to the probability they win given they exert no effort. This is the probability the other contestant puts in no effort,[20]

$$v_s^\ell(\mu_s, \mu_w) = p_2(1 - \mu_w)L_{w2}^*(0) = p_2 \left(\frac{a^h - 1}{a^h}\right)(\mu_s - \mu_w) \quad \text{and}$$
$$v_w^\ell(\mu_w, \mu_s) = p_2(1 - \mu_s)L_{s2}^*(0) = 0.$$

Proof of Proposition 1

In the second contest, for a given pair of beliefs, contestants will expect the following payoffs:

$$v_i^h(\mu_i, \mu_{-i}) = p_2 \left(\frac{a^h - 1}{a^h}\right)(1 - \min\{\mu_i, \mu_{-i}\}) \text{ and}$$
$$v_i^\ell(\mu_i, \mu_{-i}) = \begin{cases} p_2 \left(\frac{a^h-1}{a^h}\right)(\mu_i - \mu_{-i}), & \text{if } \mu_i \geq \mu_{-1} \\ 0, & \text{otherwise} \end{cases}.$$

For a high ability contestant believed to be high ability with probability μ_i and with opponent's belief distribution, $F_{\mu_{-i}}$, the expected payoff in the second contest is

$$\mathbb{E}_{\mu_{-i}}[v_i^h(\mu_i, \mu_{-i})] = \int_0^1 p_2 \left(\frac{a^h - 1}{a^h}\right)(1 - \min\{\mu_i, \mu_{-i}\})dF_{\mu_{-i}}(\mu_{-i}).$$

As the opponent believes the contestant is stronger, the change in expected payoff is

$$\frac{\partial}{\partial \mu_i}\mathbb{E}_{\mu_{-i}}[v_i^h(\mu_i, \mu_{-i})] = -p_2 \left(\frac{a^h - 1}{a^h}\right)(1 - F_{\mu_{-i}}(\mu_i)).$$

For a low ability contestant, the expected payoff is

$$\mathbb{E}_{\mu_{-i}}[v_i^\ell(\mu_i, \mu_{-i})] = \int_0^{\mu_i} p_2 \left(\frac{a^h - 1}{a^h}\right)(\mu_i - \mu_{-i})dF_{\mu_{-i}}(\mu_{-i}),$$

[20] Here we technically are assuming the contestant wins all ties at zero, but if the agent exerts a tiny amount of effort and we let that effort shrink to zero, then this is the payoff of the agent in the limit. Since the payoffs are continuous, these limits must be the payoffs of the low ability contestants.

with change in expected payoff given by

$$\frac{\partial}{\partial \mu_i}\mathbb{E}_{\mu_{-i}}[v_i^{\ell}(\mu_i,\mu_{-i})] = p_2\left(\frac{a^h-1}{a^h}\right)F_{\mu_{-i}}(\mu_i).$$

For all $\mu_i \in (\underline{M}_{-i}, \overline{M}_{-i})$, $\frac{\partial}{\partial \mu_i}\mathbb{E}_{\mu_{-i}}[v_i^h(\mu_i,\mu_{-i})] < 0$ and $\frac{\partial}{\partial \mu_i}\mathbb{E}_{\mu_{-i}}[v_i^{\ell}(\mu_i,\mu_{-i})] > 0$.

Proof of Lemma 2

Let $x, x' \in X_{i1}$ such that $x < x'$ and $\mu(x) > \mu(x')$. Then $0 \le \mu(x') < \mu(x) \le 1$ which implies $x \in X_{i1}^h \subseteq BR_{i1}^h(\sigma_{-i})$ and $x' \in X_{i1}^{\ell} \subseteq BR_{i1}^{\ell}(\sigma_{-i})$. Best responses require

$$\begin{aligned}
p_1\mathbb{E}[w_i(x',x_{-i1})] - x' + \mathbb{E}[v_i^{\ell}(\mu(x'),\mu(x_{-i1}))]& \\
\ge p_1\mathbb{E}[w_i(x,x_{-i1})] - x + \mathbb{E}[v_i^{\ell}(\mu(x),\mu(x_{-i1}))],&\text{ and}\\
p_1\mathbb{E}[w_i(x',x_{-i1})] - \frac{x'}{a^h} + \mathbb{E}[v_i^{h}(\mu(x'),\mu(x_{-i1}))]& \\
\le p_1\mathbb{E}[w_i(x,x_{-i1})] - \frac{x}{a^h} + \mathbb{E}[v_i^{h}(\mu(x),\mu(x_{-i1}))].&
\end{aligned}$$

If follows that

$$\begin{aligned}
p_1(\mathbb{E}[w_i(x',x_{-i1})] - \mathbb{E}[w_i(x,x_{-i1})]) + \mathbb{E}[v_i^{\ell}(\mu(x'),\mu(x_{-i1}))] - \mathbb{E}[v_i^{\ell}(\mu(x),\mu(x_{-i1}))]& \\
\ge x' - x, \text{ and}& \\
p_1(\mathbb{E}[w_i(x',x_{-i1})] - \mathbb{E}[w_i(x,x_{-i1})]) + \mathbb{E}[v_i^{h}(\mu(x'),\mu(x_{-i1}))] - \mathbb{E}[v_i^{h}(\mu(x),\mu(x_{-i1}))]& \\
\le \frac{x'-x}{a^h}.&
\end{aligned}$$

From Proposition 1, $\mu(x) > \mu(x')$ implies

$$\begin{aligned}
\mathbb{E}[v_i^{\ell}(\mu(x'),\mu(x_{-i1}))] - \mathbb{E}[v_i^{\ell}(\mu(x),\mu(x_{-i1}))] &\le 0,\\
\text{and } \mathbb{E}[v_i^{h}(\mu(x'),\mu(x_{-i1}))] - \mathbb{E}[v_i^{h}(\mu(x),\mu(x_{-i1}))] &\ge 0.
\end{aligned}$$

Combining the previous inequalities,

$$x' - x \le p_1(\mathbb{E}[w_i(x',x_{-i1})] - \mathbb{E}[w_i(x,x_{-i1})]) \le \frac{x'-x}{a^h},$$

which cannot be true given $a^h > 1$.

Proof of Proposition 2

Equilibrium conditions are satisfied when $H_1^*(x) = L_1^*(x) = 1$ for $x \ge 0$ and 0 otherwise (i.e. the output densities of both high and low ability contestants consist of a single mass point at $x = 0$), $\mu^*(x) = \hat{\mu}$ for $x \ge 0$, and second period distribution functions are as characterized in (5).

To show that there can be no equilibrium where $\tilde{x} \in X_{i1}$, such that $\tilde{x} > 0$, assume that there is. Then $\tilde{x} \in BR_{i1}^{\ell}(\sigma_{-i}) \cup BR_{i1}^{h}(\sigma_{-i})$. If $\tilde{x} \in BR_{i1}^{\ell}(\sigma_{-i})$ then $\mathbb{E}[v_i^{\ell}(\mu(\tilde{x}),\mu(x_{-i1}))]-$

$\mathbb{E}[v_i^\ell(\mu(0), \mu(x_{-i1}))] \geq \tilde{x} > 0$ which implies that $\mu(\tilde{x}) > \mu(0) \geq 0$. Because $\mu(\tilde{x}) > 0$, equilibrium conditions on the belief function require that $\tilde{x} \in X_{i1}^h \subset BR_{i1}^h$ and therefore $\mathbb{E}[v_i^h(\mu(\tilde{x}), \mu(x_{-i1}))] - \mathbb{E}[v_i^h(\mu(0), \mu(x_{-i1}))] \geq \frac{\tilde{x}}{a^h} > 0$, which cannot be true when $\mu(\tilde{x}) > \mu(0)$, a contradiction.

If $\tilde{x} \in BR_{i1}^h(\sigma_{-i}) \setminus BR_{i1}^\ell(\sigma_{-i})$, then $\mu(\tilde{x}) < \mu(0)$ which implies by Lemma 2 that $0 \notin X_{i1}$. This however would require the existence of a positive output in $BR_{i1}^\ell(\sigma_{-i})$, which we just ruled out.

Proof of Lemma 3

To show the output distributions are continuous, it is sufficient to show that no outputs are produced with positive probability. In a symmetric equilibrium, if an output is played with positive probability by one type of contestant, then it must be played with positive probability by both contestants of this type. Let $\tilde{x} \in \{X_{i1}^\ell \cup X_{i1}^h\}$ be played with probability $q > 0$. Then

$$\mathbb{E}[w_i(\tilde{x}, x_{-i1})] + \frac{q}{2} \leq \mathbb{E}[w_i(x, x_{-i1})] \text{ for all } x > \tilde{x}.$$

Since, $\tilde{x} \in BR_{i1}^\theta(\sigma_{-i})$, for some θ, then for all $x \geq 0$,

$$\begin{aligned} p_1\mathbb{E}[w_i(\tilde{x}, x_{-i1})] - \frac{\tilde{x}}{a^\theta} + \mathbb{E}[v_i^\theta(\mu_i(\tilde{x}), \mu_{-i})] \\ \geq p_1\mathbb{E}[w_i(x, x_{-i1})] - \frac{x}{a^\theta} + \mathbb{E}[v_i^\theta(\mu_i(x), \mu_{-i})]. \end{aligned}$$

Combining the above inequalities,

$$p_1\frac{q}{2} \leq \mathbb{E}[v_i^\theta(\mu_i(\tilde{x}), \mu_{-i})] - \mathbb{E}[v_i^\theta(\mu_i(x), \mu_{-i})] + \frac{x - \tilde{x}}{a^\theta}.$$

For all $x \in (\tilde{x}, \tilde{x} + \varepsilon)$, where $\varepsilon = \frac{a^\theta p_1 q}{2}$ we have

$$\mathbb{E}[v_i^\theta(\mu_i(\tilde{x}), \mu_{-i})] - \mathbb{E}[v_i^\theta(\mu_i(x), \mu_{-i})] > 0. \tag{27}$$

From Proposition 1, if $\theta = \ell$, then $\mu_i(\tilde{x}) > \mu_i(x)$ and $\tilde{x} \in X_{i1}^\ell \cap X_{i1}^h$. Similarly, if $\theta = h$, then $\mu_i(\tilde{x}) < \mu_i(x)$ and $\tilde{x} \in X_{i1}^\ell \cap X_{i1}^h$. In either case, $\tilde{x} \in BR_{i1}^\ell(\sigma_{-i}) \cap BR_{i1}^h(\sigma_{-i})$. However, (27) cannot hold for both $\theta = \ell$ and $\theta = h$, a contradiction.

Proof of Lemma 4

From Lemma 3 we now can use the fact that $L_1^*(x)$ and $H_1^*(x)$, and therefore $F_1^*(x)$, are continuous in x and we have that in equilibrium $\mathbb{E}[w_i(x, x_{-i1})] = \Pr(x_{-i1} < x | \sigma_{-i}^*) = \Pr(x_{-i1} \leq x | \sigma_{-i}^*) = F_1^*(x)$. Combined with Lemma 2, we have $\Pr(\mu^*(x_{-i1}) < \mu^*(x) | \sigma_{-i}^*) \leq \mathbb{E}[w_i(x, x_{-i1})] = F_1^*(x) \leq \Pr(\mu^*(x_{-i1}) \leq \mu^*(x) | \sigma_{-i}^*) = F_{\mu_{-i}}(\mu^*(x))$. The proof follows in four steps.

(1) We first show that $x_{\ell *} = 0$. We do this by first showing that $x_{\ell *} \leq x_{h*}$, and then showing that $x_{\ell *}$ cannot be larger than zero.

Let $x_{h*} < x_{\ell *}$. Since $x_{h*} = \inf X_{i1}^h$, $\forall \varepsilon > 0$, $\exists x_\varepsilon$ such that $x_{h*} \leq x_\varepsilon < x_{h*} + \varepsilon$ and $x_\varepsilon \in X_{i1}^h$. In particular, this holds for $\varepsilon^* = x_{\ell *} - x_{h*}$. Then $x_{\varepsilon^*} \in \{X_{i1}^h \setminus X_{i1}^\ell\}$ and $\mu^*(x_{\varepsilon^*}) = 1$. However, from Lemma 2 we would have $\mu^*(x) = 1$ for all $x \in X_{i1}^\ell$, which cannot hold. Therefore $x_{h*} \geq x_{\ell *}$.

If $0 < x_{\ell *} < x_{h*}$, then by Lemma 3, $\exists \delta$ with $0 < \delta < x_{h*} - x_{\ell *}$ such that $\forall x \in (x_{\ell *}, x_{\ell *} + \delta)$ we have $|p_1(F_1^*(x) - F_1^*(0))| = |p_1(F_1^*(x) - F_1^*(x_{\ell *}))| < x_{\ell *}$. Let $x_\delta \in X_1^\ell \cap (x_{\ell *}, x_{\ell *} + \delta)$. Then $\mu(x_\delta) = 0$ and $p_1(F_1^*(x_\delta) - F_1^*(0)) < x_\delta$. However this implies

$$p_1 F_1^*(0) + \mathbb{E}[v_i^\ell(\mu(0), \mu_{-i})] > p_1 F_1^*(x_\delta) + \mathbb{E}[v_i^\ell(\mu(x_\delta), \mu_{-i})] - x_\delta,$$

and therefore $x_\delta \notin BR_{i1}^\ell(\sigma_{-i})$, a contradiction.

If $0 < x_{\ell *} = x_{h*}$, then $\exists x_\ell, x_h$ such that $x_\ell \leq x_h$, $x_\ell \in X_{i1}^\ell$, $x_h \in X_{i1}^h$, and $p_1(F_1^*(x_\ell) - F_1^*(x_{\ell *})) = p_1 F_1^*(x_\ell) < x_{\ell *} < x_\ell$ and $p_1(F_1^*(x_h) - F_1^*(x_{h*})) = p_1 F_1^*(x_h) < x_{h*}/a_h < x_h/a_h$, by the continuity of $F_1^*(x)$. It follows that $x_\ell \in X_{i1}^\ell$ implies

$$p_1 F_1^*(x_\ell) - x_\ell + E[v_i^\ell(\mu(x_\ell), \mu_{-i})] \geq p_1 F_1^*(0) + \mathbb{E}[v_i^\ell(\mu(0), \mu_{-i})]$$

and $\mathbb{E}[v_i^\ell(\mu(x_\ell), \mu_{-i})] > \mathbb{E}[v_i^\ell(\mu(0), \mu_{-i})]$, which requires $\mu(x_\ell) > \mu(0)$. Similarly, $x_h \in X_{i1}^h$ implies that $\mu(x_h) < \mu(0)$. Combining these two inequalities leads to $\mu(x_h) < \mu(x_\ell)$. This contradicts Lemma 2. Therefore we must have $0 = x_{\ell *} \leq x_{h*}$.

(2) We next show that $x_{h*} \leq x_\ell^*$.

If $x_\ell^* < x_{h*}$, then $\forall x \in (x_\ell^*, x_{h*})$, $x \notin \{X_{i1}^\ell \cup X_{i1}^h\}$. Let $\tilde{x} = \frac{x_\ell^* + x_{h*}}{2}$ and $\varepsilon = x_{h*}/a^h - \tilde{x}/a^h$. There is a $\delta > 0$ such that $\forall x \in (x_{h*}, x_{h*} + \delta)$, $p_1(F_1^*(x) - F_1^*(x_{h*})) < \varepsilon$. Pick an x_δ such that $x_\delta \in (x_{h*}, x_{h*} + \delta)$ and $x_\delta \in X_{i1}^h$. Then $p_1(F_1(x_\delta) - F_1(x_{h*})) = p_1(F_1(x_\delta) - F_1(x')) < \varepsilon$, $x_\delta/a^h - \tilde{x}/a^h > \varepsilon$, and $\mathbb{E}[v_i^h(\mu(x_\delta), \mu_{-i})] \leq \mathbb{E}[v_i^h(\mu(\tilde{x}), \mu_{-i})]$. Then

$$p_1 F_1^*(\tilde{x}) + \mathbb{E}[v_i^h(\mu(\tilde{x}), \mu_{-i})] - \frac{\tilde{x}}{a^h} > p_1 F_1^*(x_\delta) + \mathbb{E}[v_i^h(\mu(x_\delta), \mu_{-i})] - \frac{x_\delta}{a^h},$$

a contradiction. So we can conclude that $x_\ell^* \leq x_{h*}$.

Also $x_\ell^* \leq x_h^*$. If we assume otherwise, then we can find $x \in \{X_{i1}^\ell \setminus X_{i1}^h\}$ where $x > x_h^*$ and $\mu(x) = 0$. Lemma 2 rules out this possibility.

We have shown so far that $0 = x_{\ell *} \leq x_{h*} \leq x_\ell^* \leq x_h^*$.

(3) For all $x \in (x_{\ell *}, x_{h*})$, $x \in BR_{i1}^\ell(\sigma_{-i})$ and for all $x \in (x_\ell^*, x_h^*)$, $x \in BR_{i1}^h(\sigma_{-i})$.

Given $x_{\ell *} < x_{h*}$, let $X_c^\ell = \{x | x \in (x_{\ell *}, x_{h*}) \setminus BR_{i1}^\ell(\sigma_{-i})\}$. If $x \in X_c^\ell$, then $\exists \varepsilon > 0$ such that for all $x' \in (x_{\ell,*}, x_{h,*}) \cap X_{i1}^\ell$,

$$p_1 F_1^*(x) + \mathbb{E}[v_i^\ell(\mu(x), \mu_{-i})] - x < p_1 F_1^*(x') + \mathbb{E}[v_i^\ell(\mu(x'), \mu_{-i})] - x' - \varepsilon,$$

where $\mathbb{E}[v_i^\ell(\mu(x), \mu_{-i})] \geq \mathbb{E}[v_i^\ell(\mu(x'), \mu_{-i})]$ as $\mu(x') = 0$. Therefore $p_1 F_1^*(x) - x < p_1 F_1^*(x') - x' - \varepsilon$, and for all $x' > x$, $p_1(F_1^*(x') - F_1^*(x)) > x' - x - \varepsilon$.

Since $F_1^*(x)$ is continuous, there is a $\delta_\varepsilon > 0$ such that for all $x' \in X_{i1}^\ell$, $|x' - x| \geq \delta_\varepsilon$.

This implies that x is contained in an interval which is a subset of X_c^ℓ. Let a and b be the infimum and supremem of this interval respectively.

- If $b < x_{h*}$, then $\exists x' < x_{h*},\ x' \in X_{i1}^\ell$ where $|x' - b| < \delta, \forall \delta > 0$. Then, by the continuity of $F_1^*(x)$, $\exists x' \in X_{i1}^\ell$ and $p_1(F_1^*(x') - F_1^*(b)) < b - \frac{a+b}{2}$. Then we know that

$$p_1 F_1^*(x') - p_1 F_1^*\left(\frac{a+b}{2}\right) < b - \frac{a+b}{2} \text{ and}$$
$$\mathbb{E}[v_i^\ell(\mu(x'), \mu_{-i})] \le \mathbb{E}\left[v_i^\ell\left(\mu\left(\frac{a+b}{2}\right), \mu_{-i}\right)\right],$$

  which contradicts $x' \in BR_{i1}^\ell(\sigma_{-i})$.

- If $b = x_{h*}$, then $\exists x' \in X_{i1}^h$, where $|x' - x_{h*}| < \delta,\ \forall \delta > 0$. We can take $x' \in X_{i1}^h$ such that $p_1(F_1^*(x') - F_1^*(x_{h*})) < \frac{b}{a^h} - \frac{a+x_{h*}}{2a^h}$.
  - If $x' \notin X_{i1}^\ell$ then $\mu(x') = 1$, but since $\mathbb{E}[v_i^h(\mu(x'), \mu_{-i})] \le \mathbb{E}\left[v_i^h\left(\mu\left(\frac{a+x_{h*}}{2}\right), \mu_{-i}\right)\right]$, then this contradicts $x' \in BR_{i1}^h(\sigma_{-i})$.
  - If $x' \in X_{i1}^\ell$, then $\mu(x') \in [0,1]$. If $\mu(x') \le \mu(\frac{a+x_{h*}}{2})$, then this contradicts $x' \in BR_{i1}^\ell(\sigma_{-i})$, but if $\mu(x') \ge \mu(\frac{a+x_{h*}}{2})$, this contradicts $x' \in BR_{i1}^h(\sigma_{-i})$.

Therefore X_c^ℓ must be empty.

Similarly, define $X_c^h = \{x | x \in (x_\ell^*, x_h^*) \setminus BR_{i1}^h(\sigma_{-i})\}$ and let $x \in X_c^h$. Then $\exists \delta_\varepsilon > 0$ such that for all $x' \in X_{i1}^h$, $|x' - x| \ge \delta_\varepsilon > 0$. Take a and b to be the infimum and supremum respectively of the interval of X_c^h containing x noting that $b < x_h^*$.

There is an $x' \in X_{i1}^h$ where $|x' - b| < \delta$ for all $\delta > 0$. Then we can take $x' \in BR_{i1}^h(\sigma_{-i})$ such that $p_1(F_1^*(x') - F_1^*(b)) < \frac{b}{a^h} - \frac{b+a}{2a^h}$. This implies $p_1(F_1^*(x') - F_1^*(\frac{b+a}{2})) < \frac{x'}{a^h} - \frac{b+a}{2a^h}$ and

$$p_1 F_1^*\left(\frac{b+a}{2}\right) - \frac{b+a}{2a^h} + \mathbb{E}\left[v_i^h\left(\mu\left(\frac{b+a}{2}\right), \mu_{-i}\right)\right] > p_1 F_1^*(x') - \frac{x'}{a^h} + \mathbb{E}[v_i^h(\mu(x'), \mu_{-i})].$$

This contradicts $x' \in BR_{i1}^h(\sigma_{-i})$, and therefore X_c^h must be empty.

(4) Lastly, we show that $x_{h*} < x_\ell^*$, and for all $x \in (x_{h*}, x_\ell^*)$, $x \in BR_{i1}^\ell(\sigma_{-i}) \cap BR_{i1}^h(\sigma_{-i})$.

If $x_\ell^* = x_{h*}$, then $\forall \delta > 0$, there is $x_\ell \in X_{i1}^\ell$ and $x_h \in X_{i1}^h$ where $|x_h - x_\ell| < \delta$. By the continuity of $F_1^*(x)$, there is x_h and x_ℓ for which

$$p_1 F_1^*(x_h) - \frac{x_h}{a^h} - \left(p_1 F_1^*(x_\ell) - \frac{x_\ell}{a^h}\right) < \mathbb{E}[v_i^h(\mu(x_\ell), \mu_{-i})] - \mathbb{E}[v_i^h(\mu(x_h), \mu_{-i})]$$

since $\mu(x_\ell) = 0$, $\mu(x_h) = 1$, and $\mathbb{E}[v_i^h(0, \mu_{-i})] - \mathbb{E}[v_i^h(1, \mu_{-i})] > 0$. Then

$$p_1 F_1^*(x_\ell) - \frac{x_\ell}{a^h} + \mathbb{E}[v_i^h(\mu(x_\ell), \mu_{-i})] > p_1 F_1^*(x_h) - \frac{x_h}{a^h} + \mathbb{E}[v_i^h(\mu(x_h), \mu_{-i})],$$

which contradicts $x_h \in BR^h_{i1}(\sigma_{-i})$.

Define $X_c = \{x | x \in (x_{h*}, x^*_\ell) \setminus (BR^\ell_{i1}(\sigma_{-i}) \cup BR^h_{i1}(\sigma_{-i}))\}$. From Lemma 2, we know that for all $x' \in \{(x_{h*}, x^*_\ell) \cap (X^\ell_{i1} \cup X^h_{i1})\}$, $\mu(x') \in (0,1)$ as $\mu(x') = 1$, implies $x^*_\ell \leq x'$ and $\mu(x') = 0$ implies $x_{h*} \geq x'$. Therefore $x' \in X^\ell_1 \cap X^h_1$.

Let $x \in X_c$ be given. Then for all $x', x'' \in \{(x_{h*}, x^*_\ell) \cap (X^\ell_{i1} \cap X^h_{i1})\}$ such that $x' < x < x''$ we must by Lemma 2 have $\mu(x') \leq \mu(x'')$. Let $\mu^* \in [\sup\{\mu(x')\}, \inf\{\mu(x'')\}]$. These are well-defined as there is at least one such x' and x''.

If $\mu(x) \geq \mu^*$ then $\mathbb{E}[v^\ell_i(\mu(x), \mu_{-i})] \geq \mathbb{E}[v^\ell_i(\mu(x'), \mu_{-i})]$ for all x' and

$$p_1 F^*_1(x') - x' + \mathbb{E}[v^\ell_i(\mu(x'), \mu_{-i})] - \varepsilon_1 > p_1 F^*_1(x) - x + \mathbb{E}[v^\ell_i(\mu(x), \mu_{-i})]$$
$$\Rightarrow p_1 F^*_1(x') - x' - \varepsilon_1 > p_1 F^*_1(x) - x.$$

By continuity of $F^*_1(x)$, $\exists \delta_1 > 0$ such that $[x - \delta_1, x] \subset X_c$.

Similarly, if $\mu(x) < \mu^*$, then $\mathbb{E}[v_i(\mu(x), \mu(x^{-i}), a_h)] \geq \mathbb{E}[v_i(\mu(x''), \mu(x^{-i}), a_h)]$ for all x'' and

$$p_1 F^*_1(x'') - \frac{x''}{a^h} - \varepsilon_2 > p_1 F^*_1(x) - \frac{x}{a^h}.$$

By continuity, $\exists \delta_2 > 0$ such that $[x, x + \delta_2] \subset X_c$. In either case, if $x \in X_c$, then there is an interval with some supremum b and infimum a such that $x \in (a, b) \subset X_c$.

If $b < x^*_\ell$, then there is an $\tilde{x} \in \{(x_{h*}, x^*_\ell) \cap X^\ell_{i1} \cap X^h_{i1}\}$ where $|\tilde{x} - b| < \delta$ for all $\delta > 0$, and therefore there is an $\tilde{x}$ where $p_1(F^*_1(\tilde{x}) - F^*_1(b)) < \frac{b}{a^h} - \frac{b+a}{2a^h}$. It follows that $p_1(F^*_1(\tilde{x}) - F^*_1(\frac{b+a}{2})) < \frac{\tilde{x}}{a^h} - \frac{b+a}{2a^h}$ and $p_1(F^*_1(\tilde{x}) - F^*_1(\frac{b+a}{2})) < \tilde{x} - \frac{b+a}{2}$.

If $\mu((b+a)/2) < \mu(\tilde{x})$ then

$$p_1 F^*_1\left(\frac{b+a}{2}\right) - \frac{b+a}{2a^h} + \mathbb{E}\left[v^h_i\left(\mu\left(\frac{b+a}{2}\right), \mu_{-i}\right)\right] > p_1 F^*_1(\tilde{x}) - \frac{\tilde{x}}{a^h} + \mathbb{E}[v^h_i(\mu(\tilde{x}), \mu_{-i})].$$

If $\mu((b+a)/2) \geq \mu(\tilde{x})$ then

$$p_1 F^*_1\left(\frac{b+a}{2}\right) - \frac{b+a}{2} + \mathbb{E}\left[v^\ell_i\left(\mu\left(\frac{b+a}{2}\right), \mu_{-i}\right)\right] > p_1 F^*_1(\tilde{x}) - \tilde{x} + \mathbb{E}[v^\ell_i(\mu(\tilde{x}), \mu_{-i})].$$

In either case, this contradicts $\tilde{x} \in X^\ell_{i1} \cap X^h_{i1}$.

If $b = x^*_\ell$, then there is an $\tilde{x} \in X^h_{i1}$, such that $|\tilde{x} - b| < \delta$, and $\mu(\tilde{x}) = 1$. This implies that $p_1(F^*_1(\tilde{x}) - F^*_1(\frac{b+a}{2})) < \frac{\tilde{x}}{a^h} - \frac{b+a}{2a^h}$, and

$$p_1 F^*_1\left(\frac{b+a}{2}\right) - \frac{b+a}{2a^h} + \mathbb{E}\left[v^h_i\left(\mu\left(\frac{b+a}{2}\right), \mu_{-i}\right)\right]$$
$$> p_1 F^*_1(\tilde{x}) - \frac{\tilde{x}}{a_h} + \mathbb{E}[v^h_i(\mu(\tilde{x}), \mu_{-i})].$$

This contradicts $\tilde{x} \in X^h_{i1}$. Therefore X_c must be empty and for all $x \in (x_{h*}, x^*_\ell)$, we must have $x \in BR^\ell_{i1}(\sigma_{-i}) \cap BR^h_{i1}(\sigma_{-i})$.

Proof of Lemma 5

To show that $\mu^*(x)$ is continuous on $(0, x_h^*)$, note that equilibrium expected payoffs of a low ability contestant are constant for all $x \in BR_{i1}^{\ell}(\sigma^*)$ and likewise for high ability contestants for all $x \in BR_{i1}^{h}(\sigma^*)$. Since $F_1^*(x)$ is continuous on $(0, \infty)$ and $\mathbb{E}[v_i^{\ell}(\mu^*(x), \mu_{-i})] = x - p_1 F_1^*(x) + K^{\ell}(p_1, p_2)$ on $[0, x_\ell^*]$, then $\mathbb{E}[v_i^{\ell}(\mu^*(x), \mu_{-i})]$ must be continuous on this interval. Similarly, $\mathbb{E}[v_i^{h}(\mu^*(x), \mu_{-i})]$ is continuous on $[x_{h*}, x_h^*]$. Since $\mathbb{E}[v_i^{h}(\mu^*(x), \mu_{-i})]$ is strictly decreasing in $\mu^*(x)$, and $\mathbb{E}[v_i^{\ell}(\mu^*(x), \mu_{-i})]$ is strictly increasing in $\mu^*(x)$, then $\mu^*(x)$ must also be continuous on $BR_{i1}^{\ell}(\sigma^*) \cup BR_{i1}^{h}(\sigma^*) = [0, x_h^*]$.

To show the remaining properties of the equilibrium belief function, we first show that the set $[0, x_h^*] \setminus X_{i1}$ has no interior, i.e. there can be no interval $[a, b] \subset [0, x_h^*]$ where for all $x \in [a, b]$, $x \notin X_{i1}$. This implies that X_{i1} is dense in $[0, x_h^*]$.

If we let $[\tilde{a}, \tilde{b}] \subset [0, x_h^*] \setminus X_{i1}$ be given, then define a and b to be the infimum and supremum respectively of the interval in $[0, x_h^*] \setminus X_{i1}$ which contains $[\tilde{a}, \tilde{b}]$. Neither x_{h*} nor x_ℓ^* can be contained in the interval as they are the limit point of a subset of X_{i1}. Then the interval $[a, b]$ must be contained within either $[0, x_{h*}]$, $[x_{h*}, x_\ell^*]$, or $[x_\ell^*, x_h^*]$.

1. If $[a, b] \subset [0, x_{h*}]$, then for all $x \in [a, b]$, $x \in BR_{i1}^{\ell}(\sigma^*)$ and $F_1^*(x) = F_1^*(a)$. Therefore, $\mathbb{E}[v_i^{\ell}(\mu^*(b), \mu_{-i})] - b = \mathbb{E}[v_i^{\ell}(\mu^*(a), \mu_{-i})] - a$, which implies that $\mu^*(b) > \mu^*(a)$. Since $\mu^*(x)$ is continuous, then for all $\delta > 0$, there is an $x \in X_{i1}^h$ such that $|x - b| < \delta$ and $\mu^*(x) > 0$. If $x \in X_{i1}^h \setminus X_{i1}^{\ell}$, then $\mu^*(x)$=1, and $x \notin BR_{i1}^{h}(\sigma^*)$, a contradiction. If $x \in X_{i1}^h \cap X_{i1}^{\ell}$ then depending on the value of $\mu^*((a + b)/2)$, it must be that either $x \notin BR_{i1}^{h}(\sigma^*)$ or $x \notin BR_{i1}^{\ell}(\sigma^*)$, again a contradiction.

2. If $[a, b] \subset [x_{h*}, x_\ell^*]$, then for all $x \in [a, b]$, $x \in \{BR_{i1}^{\ell}(\sigma^*) \cap BR_{i1}^{h}(\sigma^*)\}$ which implies

$$\mathbb{E}[v_i^{\ell}(\mu^*(b), \mu_{-i})] - b = \mathbb{E}[v_i^{\ell}(\mu^*(a), \mu_{-i})] - a \quad \text{and}$$
$$\mathbb{E}[v_i^{h}(\mu^*(b), \mu_{-i})] - b/a^h = \mathbb{E}[v_i^{h}(\mu^*(a), \mu_{-i})] - a/a^h.$$

However, rearranging these equations, it is clear they cannot hold at the same time as the right hand sides are both strictly positive which contradicts Proposition 1:

$$\mathbb{E}[v_i^{\ell}(\mu^*(b), \mu_{-i})] - \mathbb{E}[v_i^{\ell}(\mu^*(a), \mu_{-i})] = b - a \quad \text{and}$$
$$\mathbb{E}[v_i^{h}(\mu^*(b), \mu_{-i})] - \mathbb{E}[v_i^{h}(\mu^*(a), \mu_{-i})] = \frac{b - a}{a^h}.$$

3. If $[a, b] \subset [x_\ell^*, x_h^*]$, then for all $x \in [a, b]$, $x \in BR_{i1}^{h}(\sigma^*)$ and therefore,

$$\mathbb{E}[v_i^{h}(\mu^*(b), \mu_{-i})] - b/a^h = \mathbb{E}[v_i^{h}(\mu^*(a), \mu_{-i})] - a/a^h$$

and $\mu^*(b) < \mu^*(a) \leq 1$. Then for all $\delta > 0$, there is an $x \in X_{i1}^h$ such that $|x - b| < \delta$ and $\mu^*(x) = 1$. However, this contradicts the continuity of $\mu^*(x)$.

Now, if $x \in [0, x_{h*})$ and $\mu^*(x) = \varepsilon > 0$, then by the continuity of $\mu^*(x)$, $\exists \delta > 0$ where $\forall x', |x' - x| < \delta$, $\mu^*(x) > \varepsilon/2$. However for all $\delta > 0$ there is an $x' \in X_{i1}^{\ell} \setminus X_{i1}^h$ for which $\mu^*(x') = 0$, a contradiction. Therefore $\mu^*(x) = 0$ for all $x \in [0, x_{h*})$. Note that $\mu^*(x_{h*}) = 0$ when $x_{h*} > 0$, which follows from continuity from the left. Similarly, $\mu^*(x) = 1$ for all $x \in [x^*_\ell, x^*_h]$ when $x^*_\ell < x^*_h$. To show that $\mu^*(x)$ is weakly increasing on $[x_{h*}, x^*_\ell]$, let $x, y \in [x_{h*}, x^*_\ell]$ be such that, $\mu^*(x) > \mu^*(y)$ and $x < y$. Then there is an x' and y' arbitrarily close to x and y respectively, where $x', y' \in X_{i1}$ and therefore $\mu^*(x') \leq \mu^*(y')$. This is not consistent with $\mu^*(x)$ being continuous on $[0, x^*_h]$, a contradiction.

Proof of Theorem 1

There are up to three distinct intervals in each equilibrium. We will show that the end-points of these intervals and the distribution functions on the intervals are completely determined by the first order conditions of the contestants.

Conditions for x being in $BR^h_{i1}(\sigma^*)$ and $BR^\ell_{i1}(\sigma^*)$ are[21]

$$BR^h_{i1}(\sigma^*): \ p_1 F^*_1(x) + \mathbb{E}[v^h_i(\mu^*(x), \mu_{-i})] - \frac{x}{a^h} = K^h(p_1, p_2) \ \text{ and}$$
$$BR^\ell_{i1}(\sigma^*): \ p_1 F^*_1(x) + \mathbb{E}[v^\ell_i(\mu^*(x), \mu_{-i})] - x = K^\ell(p_1, p_2) = 0.$$

For all values of $p_1 > 0$ and $p_2 > 0$, Lemma 4 shows that $x_{h*} < x^*_\ell$, and therefore the interval $[x_{h*}, x^*_\ell]$ is non-trivial. On this interval, $x \in X^\ell_{i1} \cup X^h_{i1}$ implies $x \in X^\ell_{i1} \cap X^h_{i1} \subset BR^\ell_{i1}(\sigma^*) \cap BR^h_{i1}(\sigma^*)$. Subtracting the condition for $BR^\ell_{i1}(\sigma^*)$ from the condition for $BR^h_{i1}(\sigma^*)$ gives

$$\mathbb{E}[v^h_i(\mu^*(x), \mu_{-i})] - \mathbb{E}[v^\ell_i(\mu^*(x), \mu_{-i})] = \frac{x}{a^h} - x + K^h(p_1, p_2).$$

Taking the derivative of each side with respect to output recovers (13):

$$\frac{d\mu^*(x)}{dx} \frac{\left(a^h - 1\right) p_2}{a^h} = \frac{a^h - 1}{a^h}.$$

Note that on this interval, $\frac{d\mu^*(x)}{dx} > 0$ and therefore, $F_\mu(\mu^*(x)) = F^*_1(x)$.

Taking the derivative of the condition for X^ℓ_{i1} and combining with (13) we recover (14):

$$p_1 f^*_1(x) = 1 - \frac{a^h - 1}{a^h} F^*_1(x).$$

From continuity of $F^*_1(x)$, $p_1 F^*_1(x_{h*}) = c(x_{h*})$. For a given x_{h*}, using the Picard - Lindelöf Theorem[22], we know that there is a unique solution for $f^*_1(x)$ on $[x_{h*}, x^*_\ell]$, and therefore $F^*_1(x)$ is determined on this interval.

To see why only one such x_{h*} can lead to an equilibrium, consider a different initial condition, $p_1 \tilde{F}^*(\tilde{x}_{h*}) = c(\tilde{x}_{h*})$ where $\tilde{x}_{h*} > x_{h*}$ and the associated $\tilde{f}_1(x)$ on $[\tilde{x}_{h*}, \tilde{x}^*_\ell]$. Then both $\tilde{F}^*_1(\tilde{x}_{h*}) > F^*_1(\tilde{x}_{h*})$ and $\tilde{\mu}^*(\tilde{x}_{h*}) < \mu^*(\tilde{x}_{h*})$, and for all $x \in [\tilde{x}_{h*}, x^*_\ell]$,

[21]The expected payoff of the low ability contestant is zero, as choosing $x = 0$ in the first contest is always in the best response set. This output guarantees the contestant zero payoff in the first contest and the weak position entering the second contest leading to a zero expected payoff in the second contest.

[22]The right hand side of equation (17) is continuous in x and uniformly Lipshitz continuous in $F^*_1(x)$ on the interval of $[x_{h*}, x^*_\ell]$. Also, due to the properties of the cost function, the distribution function is bounded between 0 and 1.

$\tilde{F}_1^*(x) > F_1^*(x)$, $\tilde{f}_1^*(x) < f_1^*(x)$, and $\mu^*(x) > \tilde{\mu}^*(x)$. This implies that $\tilde{H}_1^*(x_\ell^*) = \int_0^{x_\ell^*} \tilde{\mu}^*(x)\tilde{f}(x)dx < \int_0^{x_\ell^*} \mu^*(x)f(x)dx = H_1^*(x_\ell^*)$ and therefore $\tilde{L}_1^*(x_\ell^*) > L_1^*(x_\ell^*) = 1$, a contradiction. Similarly, there cannot be an additional equilibrium where $\tilde{x}_{h*} < x_{h*}$.

The belief function on this interval is determined up to a constant by equation (13). The constant is determined by $\mu^*(x_{h,*})$ which is 0 when $x_{h*} > 0$, and needs to be characterized in equilibrium when $x_{h*} = 0$. Given this constant, the equilibrium strategies of high ability and low ability contestants can be constructed on this interval.

For small values of p_1 relative to p_2, this is the only non-empty interval: $x_{h*} = 0$ and $x_\ell^* = x_h^*$. In this case, $\mu^*(x_{h*}) \in [0, \hat{\mu}]$ and $\mu^*(x_h^*) \in [\hat{\mu}, 1]$ both need to be determined in equilibrium along with x_h^*. By an argument similar to that for showing x_{h*} is unique, if $x_{h*} = 0$ then $\mu^*(x_{h*})$ is also uniquely determined. Then $\mu^*(x)$ and $F_1^*(x)$ are uniquely determined on this interval, and therefore x_h^* and $\mu^*(x_h^*)$ are also uniquely determined.

For larger p_1, $x_{h*} > 0$ and/or $x_h^* > x_\ell^*$. When the intervals are non-empty, then the belief functions on these intervals are characterized in Lemma 5. Characterization of the output distributions directly follow. For $x \in [0, x_{h*})$, $\mathbb{E}[v_i^\ell(\mu^*(x), \mu_{-i})] = 0$ as $\mu^*(x) = 0$, and therefore $p_1 F_1^*(x) = x$. For all $x \in [x_\ell^*, x_h^*]$, $\mathbb{E}[v_i^h(\mu^*(x), \mu_{-i})] = \mathbb{E}[v_i^h(1, \mu_{-i})]$ and $p_1 F_1^*(x) + \mathbb{E}[v_i^h(1, \mu_{-i})] = x/a^h + K^h(p_1, p_2)$.

Given $F_1^*(x)$ and $\mu^*(x)$ on $[0, x^*]$, the output distribution of both the low and high ability contestants can be determined. Therefore $F_1^*(x), L_1^*(x)$ and $H_1^*(x)$ are uniquely characterized on X_{i1} where $\overline{X}_{i1} = [0, x_h^*]$. These distributions along with the second period output distributions $L_2^*(x|\eta_2)$ and $H_2^*(x|\eta_2)$ form the unique symmetric perfect Bayesian equilibrium. The strategies and endpoints of the best response sets are characterized for $\hat{\mu} \in (0, 1)$, $a^h > 1$ and $p_1, p_2 > 0$ in Appendix A.

Proof of Lemma 6

The best response set of the low ability contestant always contains zero. The payoff in the first contest for this output choice is zero. Because the belief function is weakly increasing in output, choosing zero will ensure that the contestant enters the second contest in the weak (or an identical) position. In these cases, the expected payoff of a low ability contestant in the second contest is zero.

The output, x_{h*}, is in the best response set of the high ability player. Then two-period payoffs are

$$\begin{aligned} K^h(p_1, p_2) &= p_1 F_1^*(x_{h*}) - \frac{x_{h*}}{a^h} + \mathbb{E}[v_i^h(\mu(x_{h*}), \mu_{-i})] \\ &= \frac{a^h - 1}{a^h} \left(x_{h*} + p_2(1 - \mu(x_{h*})) \right). \end{aligned}$$

Proof of Lemma 7

Letting $E^\theta(p_1, p_2)$ denote the expected effort of a contestant of ability a^θ, the ex-interim payoffs of each contestant are

$$\begin{aligned} 0 &= p_1 W_1^\ell(p_1, p_2) + p_2 W_2^\ell(p_1, p_2) - E^\ell(p_1, p_2), \text{ and} \\ K^h(p_1, p_2) &= p_1 W_1^h(p_1, p_2) + p_2 W_2^h(p_1, p_2) - E^h(p_1, p_2). \end{aligned}$$

Noting that the ex-ante probability of a contestant winning each contest is 1/2 so that

$$\frac{1}{2} = \hat{\mu} W_t^h(p_1, p_2) + (1 - \hat{\mu}) W_t^\ell(p_1, p_2),$$

and the expected output can be written as

$$\begin{aligned} Y(p_1, p_2) &= a^h \hat{\mu} E^h(p_1, p_2) + (1 - \hat{\mu}) E^\ell(p_1, p_2) \\ &= a^h \hat{\mu} \left(p_1 W_1^h(p_1, p_2) + p_2 W_2^h(p_1, p_2) - K^h(p_1, p_2) \right) \\ &\quad + (1 - \hat{\mu}) \left(p_1 W_1^\ell(p_1, p_2) + p_2 W_2^\ell(p_1, p_2) \right) \\ &= \frac{p_1 + p_2}{2} + (a^h - 1)\hat{\mu} \left(p_1 W_1^h(p_1, p_2) + p_2 W_2^h(p_1, p_2) \right) - a^h \hat{\mu} K^h(p_1, p_2). \end{aligned}$$

Proof of Lemma 8

Belief distributions that arise after the first contest for different prize structures must be equal at least at one point between 0 and 1. Otherwise, if the distributions do not cross then one distribution FOSD the other and the distributions cannot have the same expected value. However, the expectation of the probability that a contestant is high ability is $\hat{\mu}$ in either case.

Let $\tilde{\mu}(\hat{\tilde{x}}) = \mu(\hat{x}) = \hat{M} \in (0, 1)$ be a point of intersection for belief distributions $\tilde{F}_\mu^*(M)$ and $F_\mu^*(M)$. Note that

$$f_\mu^*(\hat{M}) = \frac{\partial}{\partial \mu} F_1^*(\mu^{*-1}(\hat{M})) = \frac{f_1^*(\mu^{*-1}(\hat{M}))}{\mu^{*\prime}(\mu^{*-1}(\hat{M}))} = \frac{f_1^*(\hat{x})}{\mu^{*\prime}(\hat{x})}.$$

From equations (13) and (14),

$$\frac{f_1^*(\hat{x})}{\mu^{*\prime}(\hat{x})} = \frac{p_2}{p_1} \left(1 - \frac{a^h - 1}{a^h} F_1^*(\hat{x}) \right).$$

Because $\tilde{F}_1^*(\tilde{\mu^*}^{-1}(\hat{M})) = F_1^*(\mu^{*-1}(\hat{M}))$, then $\tilde{f}_\mu^*(\hat{M}) \lesseqgtr f_\mu^*(\hat{M})$ when $\frac{\tilde{p}_2}{\tilde{p}_1} \lesseqgtr \frac{p_2}{p_1}$. Given that $\tilde{f}_\mu^*(\hat{M}) < f_\mu^*(\hat{M})$, as in the case when first contest prize is increased for a fixed second contest prize, this implies that $\tilde{F}_\mu^*(M)$ crosses $F_\mu^*(M)$ exactly once from above and $\tilde{F}_\mu^*(M) <_{SOSD} F_\mu^*(M)$. An increase the second contest prize for fixed first contest prize implies $\tilde{F}_\mu^*(M)$ crosses $F_\mu^*(M)$ exactly once from below and $\tilde{F}_\mu^*(M) >_{SOSD} F_\mu^*(M)$.

Proof of Proposition 3

The probability that a low ability player wins the first stage is $2(1-\hat{\mu})W_1^\ell(p_1,p_2)$ where

$$\begin{aligned} W_1^\ell(p_1,p_2) &= \int_0^1 \ell_\mu^*(M)F_\mu^*(M)dM + \frac{1}{2}L_\mu^*(0)F_\mu^*(0) \\ &= \frac{1}{1-\hat{\mu}}\left(\int_0^1 f_\mu^*(M)F_\mu^*(M)dM - \int_0^1 Mf_\mu^*(M)F_\mu^*(M)dM + \frac{1}{2}F^*(0)^2\right) \\ &= \frac{1}{2(1-\hat{\mu})}\int_0^1 (F_\mu^*(M))^2 dM, \end{aligned}$$

and where the last equality follows from integration by parts. Whenever $\tilde{F}_\mu^* >_{SOSD} F_\mu^*$ then $\int_0^1 (F_\mu^*(M))^2 dM < \int_0^1 (\tilde{F}_\mu^*(M))^2 dM$ which follows from

$$\begin{aligned} \int_0^1 (\tilde{F}_\mu^*(M))^2 - (F_\mu^*(M))^2 dM &= \int_0^1 (\tilde{F}_\mu^*(M) + F_\mu^*(M))(\tilde{F}_\mu^*(M) - F_\mu^*(M))dM \\ &> (\tilde{F}_\mu^*(\hat{M}) + F_\mu^*(\hat{M}))\int_0^1 \tilde{F}_\mu^*(M) - F_\mu^*(M)dM \\ &= (\tilde{F}_\mu^*(\hat{M}) + F_\mu^*(\hat{M}))(\tilde{F}_\mu^*(1^-) - F_\mu^*(1^-)) \geq 0, \end{aligned}$$

where $F_\mu^*(1^-) = 1 - f_\mu^*(1)$ is the left limit of the distribution function as $M \to 1$. The middle inequality uses the fact that $\tilde{F}_\mu(\hat{M}) = F_\mu(\hat{M})$ with $\tilde{F}_\mu(M) \lesseqgtr F_\mu(M)$ when $M \lesseqgtr \hat{M}$. The last equality follows from $\int_0^1 F_\mu(M)dM = F_\mu(1^-) - \mathbb{E}[M]$, and the final inequality follows from $\tilde{F}_\mu(1^-) \geq F_\mu(1^-)$.

In the second stage, the probability that a low ability contestant wins given beliefs (μ_s, μ_w) is $(1-\mu_s)(1-\mu_w) + \frac{(\mu_s-\mu_w)^2}{2a^h}$. The first term is the probability that both contestants are low ability, in which case a low ability contestant is guaranteed to win, while the second term is the probability of a low ability contestant beating a high ability contestant. This can only happen if the low ability contestant is weaker than the high ability contestant and both produce output in $[x_s^*, x_w^*]$, which happens with probability $\frac{(\mu_s-\mu_w)^2}{a^h}$. Given uniform density of outputs on this interval, the low ability contestant wins half of the time.

The ex-ante probability that a low ability contestant wins the second stage is

$$\begin{aligned} \mathbb{E}\left[(1-\mu_s)(1-\mu_w) + \frac{(\mu_s-\mu_w)^2}{2a^h}\right] &= \mathbb{E}\left[(1-\mu_1)(1-\mu_2) + \frac{(\mu_1-\mu_2)^2}{2a^h}\right] \\ &= (1-\hat{\mu})^2 + \frac{1}{2a^h}\mathbb{E}[(\mu_1-\mu_2)^2], \end{aligned}$$

where μ_1 and μ_2 are independent draws from $F_\mu^*(M)$. This follows as the realization of μ_w and μ_s stems from the realization of two independent draws from the expected belief distribution, μ_1, μ_2, where $\mu_s = \max\{\mu_1,\mu_2\}$ and $\mu_w = \min\{\mu_1,\mu_2\}$, and the expression is exchangeable in μ_s and μ_w.

The first term is constant for all prizes while the second term increases as p_1/p_2

increases. To see this let $p_1/p_2 > \tilde{p}_1/\tilde{p}_2$, then from Lemma 8, $F^*_\mu(M) <_{SOSD} \tilde{F}^*_\mu(M)$. Then $F^*_\mu(M)$ is a mean preserving spread of $\tilde{F}^*_\mu(M)$ and there are independent and mean zero $\varepsilon_1, \varepsilon_2$ such that

$$\begin{aligned}\mathbb{E}[(\mu_1 - \mu_2)^2] &= \mathbb{E}[(\tilde{\mu}_1 + \varepsilon_1 - (\tilde{\mu}_2 + \varepsilon_2)^2] \\ &= \mathbb{E}[(\tilde{\mu}_1 - \tilde{\mu}_2)^2] + \mathbb{E}[(\varepsilon_1 - \varepsilon_2)^2] \\ &> \mathbb{E}[(\tilde{\mu}_1 - \tilde{\mu}_2)^2].\end{aligned}$$

Proof of Proposition 4

For $p_1 + p_2 = \bar{p}$, we first show that $R^h(p_1, p_2)$ is maximized when $p_1 = \bar{p}$ and $p_2 = 0$ and $p_1 = 0$ and $p_2 = \bar{p}$. We then show that for these same prizes $K^h(p_1, p_2)$ is minimized. From Lemma 7, this implies that the output must be maximized for those sets of prizes.

Given that strategies are ex-ante symmetric, the high ability contestant has a equal chance of beating another high ability contestant in each stage for any set of prizes. Therefore, the expected revenue of the high ability contestant can never exceed $(p_1 + p_2)\left(1 - \frac{\hat{\mu}}{2}\right)$ and this revenue is only achieved when high ability contestants always beat low ability contestants in stages where there is a positive prize.

When $p_1 + p_2 = \bar{p}$, we can show this upper bound is attained at the two endpoints of the prize allocation, namely $p_1 = \bar{p}$ and $p_1 = 0$. When $p_1 = \bar{p}$ then only the first contest impacts revenues and there is a separating equilibrium where high ability contestants always beat low ability $x_{h*} = x^*_\ell$. Similarly, when $p_1 = 0$, only the second contest matters and the same separating equilibrium is played in that contest. In either case, the revenue is $\bar{p}(1 - \hat{\mu}/2)$. All other interior prize distributions lead to lower revenue. By Lemma 4, any $p_1 \in (0, \bar{p})$ leads to equilibrium best response sets where $x_{h*} < x^*_\ell$, and therefore the probability that a high ability contestant loses to a low ability contestant is positive.

When $p_1 = \bar{p}$, and $p_2 = 0$ then $x_{h*} = (1 - \hat{\mu})\bar{p}$ and $\mu^*(x_{h*}) = 0$. From Lemma 6, $K^h(\bar{p}, 0) = \frac{a^h - 1}{a^h}(1 - \hat{\mu})\bar{p}$. When $p_1 = 0$ and $p_2 = \bar{p}$, then $x_{h*} = 0$ and $\mu^*(x_{h*}) = \hat{\mu}$. Then $K^h(0, \bar{p}) = \frac{a^h - 1}{a^h}(1 - \hat{\mu})\bar{p}$. We now show that $K^h(p_1, p_2) > K^h(\bar{p}, 0)$ for all other prize allocations.

For $\frac{p_1}{p_2} \leq \frac{1 - \hat{\mu}}{\frac{a^h}{a^h - 1}\left(\frac{a^h}{a^h - 1}\log(a^h) - 1\right)} \equiv \underline{p}^h$, the best response sets of high and low ability contestants completely overlap, so that $x_{h*} = 0$, and from (22) the belief is $\mu^*(x_{h*}) = \hat{\mu} - \frac{p_1}{p_2}\frac{a^h}{a^h - 1}\left(1 - \frac{\log(a^h)}{a^h - 1}\right)$. From Lemma 6, payoffs for the high ability contestant are $K^h(p_1, p_2) = p_2\frac{a^h - 1}{a^h}(1 - \hat{\mu}) + p_1\left(1 - \frac{\log(a^h)}{a^h - 1}\right)$. Letting $p_2 = \bar{p} - p_1$ and taking the derivative

$$\frac{\partial K^h(p_1, \bar{p} - p_1)}{\partial p_1} = -\frac{a^h - 1}{a^h}(1 - \hat{\mu}) + 1 - \frac{\log(a^h)}{a^h - 1}.$$

This is a constant relative to p_1 and is increasing in $\hat{\mu}$. For $\hat{\mu} \geq 1/2$ and using the inequality $\log(x+1) \le \frac{x}{\sqrt{x+1}}$,

$$\frac{\partial K^h(p_1, \bar{p} - p_1)}{\partial p_1} \ge \frac{a^h + 1}{2a^h} - \frac{1}{\sqrt{a^h}} = \frac{(\sqrt{a^h} - 1)^2}{2a^h} \ge 0.$$

Therefore for $0 < \frac{p_1}{p_2} \le \underline{p}^h$, $K^h(p_1, \bar{p} - p_1) > K^h(\bar{p}, 0)$.

When $p_1/p_2 > \underline{p}^h$, then $\hat{\mu} \ge \frac{1}{2} > \overline{\mu}(a^h) \equiv \frac{1}{\log(a^h)} - \frac{1}{a^h - 1}$ implies from the construction in Appendix A that $\mu^*(x^*_\ell) = 1$ and $x^*_h > x^*_\ell$. Using $L^*_1(x^*_\ell) = 1$ and $\mu^*(x^*_\ell) = 1$, we have

$$1 - \hat{\mu} = \frac{a^h}{a^h - 1}\left(1 - \mu^*(x_{h*}) - \frac{p_1}{p_2}(F^*_1(x^*_\ell) - F^*_1(x_{h*}))\right).$$

Plugging in the expression for the equilibrium output distribution we recover (23),

$$\frac{p_2}{p_1}\frac{a^h - 1}{a^h}(1 - \hat{\mu}) = \frac{p_2}{p_1}(1 - \mu^*(x_{h*})) - \left(\frac{a^h}{a^h - 1} - \frac{x_{h*}}{p_1}\right)\left(1 - e^{\frac{-a^h - 1}{a^h}\frac{p_2}{p_1}(1 - \mu^*(x_{h*}))}\right).$$

Payoffs are given by

$$K^h(p_1, p_2) = \frac{a^h - 1}{a^h}(x_{h*} + p_2(1 - \mu^*(x_{h*}))) = \frac{a^h - 1}{a^h}x^*_\ell.$$

For $p_1/p_2 > \underline{p}^h$, $x_{h*} \ge 0$ and $\mu^*(x_{h*}) \ge 0$, but following Lemma 5, both cannot be positive. Three cases follow. In the edge case where $p_1/p_2 = \overline{p}^h(\hat{\mu}, a^h)$, both $x_{h*} = 0$ and $\mu^*(x_{h*}) = 0$ and equation (23) becomes

$$\frac{p_1}{p_2}\frac{a^h}{a^h - 1}\left(1 - e^{-\frac{a^h - 1}{a^h}\frac{p_2}{p_1}}\right) = 1 - \frac{a^h - 1}{a^h}(1 - \hat{\mu}). \tag{28}$$

and payoffs are $\frac{a^h - 1}{a^h}p_2$. We first show that the left hand side (LHS) of (28) is decreasing in p_2, so to lower that side of the equation, p_2 should be increased. Letting $A = \frac{a^h - 1}{a^h}$ and taking the derivative of this side with respect to $\tilde{p} = \frac{p_2}{\bar{p} - p_2}$ gives

$$\frac{\partial}{\partial \tilde{p}}\frac{(1 - e^{-\tilde{p}A})}{\tilde{p}A} = \frac{Ae^{-\tilde{p}A}}{\tilde{p}A} - \frac{1 - e^{-\tilde{p}A}}{\tilde{p}^2 A} = \frac{1 + \tilde{p}A - e^{\tilde{p}A}}{\tilde{p}^2 A e^{\tilde{p}A}} < 0.$$

The last inequality follows from $e^x > 1 + x$. Now, letting $p_2 = \bar{p}(1 - \hat{\mu})$ (and therefore $p_1 = \bar{p}\hat{\mu}$), (28) becomes

$$\frac{\hat{\mu}}{1 - \hat{\mu}}\frac{a^h}{a^h - 1}\left(1 - e^{-\frac{a^h - 1}{a^h}\frac{1 - \hat{\mu}}{\hat{\mu}}}\right) = 1 - \frac{a^h - 1}{a^h}(1 - \hat{\mu}).$$

Letting $z = \frac{\hat{\mu}}{1 - \hat{\mu}}\frac{a^h}{a^h - 1}$, this becomes $\frac{1 - e^{-z}}{z} = 1 - \frac{z}{\hat{\mu}}$, where $z \in [0, 1)$. For $\hat{\mu} \ge 1/2$, $1 - z + \frac{z^2}{\hat{\mu}} - e^{-z} \ge 1 - z + \frac{z^2}{2} - e^{-z} > 0$, which implies that if $p_2 = \bar{p}(1 - \hat{\mu})$, then LHS is larger than RHS. Therefore for (28) to hold, $p_2 > \bar{p}(1 - \hat{\mu})$ and payoffs must be larger than $K^h(\bar{p}, 0)$.

When $p_1/p_2 > \bar{p}^h$, then $\mu(x_{h*}) = 0$ and $x_{h*} > 0$. In this case (23) becomes

$$\frac{p_1 \frac{a^h}{a^h-1} - x_{h*}}{p_2}\left(1 - e^{-\frac{a^h-1}{a^h}\frac{p_2}{p_1}}\right) = 1 - \frac{a^h-1}{a^h}(1-\hat{\mu}). \tag{29}$$

Payoffs are $K^h(\bar{p} - p_2, p_2) = \frac{a^h-1}{a^h}(x_{h*} + p_2)$. When $p_2 = 0$, $x_{h*} = \bar{p}(1-\hat{\mu})$. To show payoffs are higher for $p_2 > 0$ and $p_1/p_2 > \bar{p}^h$ we show that $x_{h*} > \bar{p}(1-\hat{\mu}) - p_2$.

If $p_2 \geq \bar{p}/2$, then $x_{h*} > 0$ directly gives $x_{h*} > \bar{p}(1-\hat{\mu}) - p_2$ as $1 - \hat{\mu} < 1/2$. To show payoffs are higher for $\bar{p}/2 > p_2 > 0$ and $p_1/p_2 > \bar{p}^h$, we show that if x_{h*} is replaced with $\bar{p}(1-\hat{\mu}) - p_2$, the LHS of (29) is increasing in p_2. Because the RHS is constant in p_2 and x_{h*} and the LHS is decreasing in x_{h*} when $p_2 > 0$, this would imply that in equilibrium, $x_{h*} > \bar{p}(1-\hat{\mu}) - p_2$, and therefore $K^h(\bar{p} - p_2, p_2) > K^h(\bar{p}, 0)$.

Replacing x_{h*} with $\bar{p}(1-\hat{\mu}) - p_2$ the LHS of (29) becomes

$$\frac{(\bar{p} - p_2)\frac{a^h}{a^h-1} - (\bar{p}(1-\hat{\mu}) - p_2)}{p_2}\left(1 - e^{-\frac{a^h-1}{a^h}\frac{p_2}{p_1}}\right).$$

Letting $y = \frac{p_2}{p_1} = \frac{p_2}{\bar{p}-p_2}$, we note that $\frac{\partial}{\partial y}LHS$ and $\frac{\partial}{\partial p_2}LHS$ have the same sign. The left hand side is

$$\frac{\frac{1}{a^h-1} + \hat{\mu}(1+y)}{y}\left(1 - e^{-\frac{a^h-1}{a^h}y}\right) \propto \frac{1 + (a^h-1)\hat{\mu}(1+y)}{y}\left(1 - e^{-\frac{a^h-1}{a^h}y}\right).$$

Computing the derivative of this expression with respect to y gives

$$\begin{aligned}
\frac{\partial}{\partial y}LHS \propto{}& \frac{(a^h-1)\hat{\mu}y - (1 + (a^h-1)\hat{\mu}(1+y))}{y^2}\left(1 - e^{-\frac{a^h-1}{a^h}y}\right) \\
&+ \frac{1 + (a^h-1)\hat{\mu}(1+y)}{y}\frac{a^h-1}{a^h}e^{-\frac{a^h-1}{a^h}y} \\
&\propto y\frac{a^h-1}{a^h}(1 + (a^h-1)\hat{\mu}) + y^2\frac{(a^h-1)^2}{a^h}\hat{\mu} - ((a^h-1)\hat{\mu} + 1)\left(e^{\frac{a^h-1}{a^h}y} - 1\right) \\
&= a^h\hat{\mu}\left(y^2\left(\frac{a^h-1}{a^h}\right)^2 + 1 + y\frac{a^h-1}{a^h} - e^{\frac{a^h-1}{a^h}y}\right) - (1-\hat{\mu})\left(e^{\frac{a^h-1}{a^h}y} - 1 - \frac{a^h-1}{a^h}y\right).
\end{aligned}$$

The second line comes from multiplying the derivative by y^2 and $e^{\frac{a^h-1}{a^h}y}$. We now take $\hat{\mu} = 1/2$ because if this expression is always positive for $\hat{\mu} = 1/2$ then it is positive (and larger) for all $\hat{\mu} > 1/2$. The expression becomes

$$\begin{aligned}
&\frac{a^h}{2}\left(y^2\left(\frac{a^h-1}{a^h}\right)^2 + y\frac{a^h-1}{a^h} - e^{\frac{a^h-1}{a^h}y}\right) - \frac{1}{2}\left(e^{\frac{a^h-1}{a^h}y} - \left(1 - \frac{a^h-1}{a^h}y\right)\right) \\
&\quad\propto y^2\frac{a^h-1}{a^h} + 1 + \frac{a^h-1}{a^h}y - e^{\frac{a^h-1}{a^h}y}.
\end{aligned}$$

If this expression is greater than or equal to zero for all $a^h > 1$ and $y \in [0,1]$, then LHS of (29) is increasing in p_2 when x_{h*} is replaced by $\bar{p}(1-\hat{\mu}) - p_2$ for all $p_2 \leq \bar{p}/2$, and the desired result holds. To show this, we first note that for $a_h > 1$ and $y = 0$ the expression is 0. Fixing $a^h > 1$, then $\frac{a^h-1}{a^h}$ is fixed and between 0 and 1. We then take the derivative of this expression with respect to y and show it is positive for $y \in (0,1)$:

$$\frac{\partial}{\partial y}\, y^2\frac{a^h-1}{a^h} + 1 + \frac{a^h-1}{a^h}y - e^{\frac{a^h-1}{a^h}y} = \frac{a^h-1}{a^h}\left(2y + 1 - e^{\frac{a^h-1}{a^h}y}\right).$$

This derivative is positive for $y \in (0,1)$ as it is concave down for all y, and equals 0 when $y = 0$ and $y > 1$. The second zero being bigger than one follows from $e^{\frac{a^h-1}{a^h}} < 3$ for all $a^h > 1$.

Finally, when $\frac{p_1}{p_2} \in (\underline{p}^h, \overline{p}^h)$, then $\mu^*(x_{h*}) > 0$ and $x_{h*} = 0$, and equation (23) becomes

$$(1-\mu^*(0)) - \frac{p_1}{p_2}\frac{a^h}{a^h-1}\left(1 - e^{\frac{-a^h-1}{a^h}\frac{p_2}{p_1}(1-\mu^*(0))}\right) = \frac{a^h-1}{a^h}(1-\hat{\mu}). \tag{30}$$

We first note that the LHS of (30) is increasing in $(1-\mu^*(0))$ for fixed p_1, p_2, a^h and $\hat{\mu}$,

$$\frac{\partial LHS}{\partial(1-\mu^*(0))} = 1 - e^{-\frac{a^h-1}{a^h}\frac{p_2}{p_1}(1-\mu^*(0))} > 0.$$

Then if we replace $(1-\mu^*(0))p_2$ with $\bar{p}(1-\hat{\mu})$ in (30) it follows that $K^h(\bar{p}-p_2, p_2) = \frac{a^h-1}{a^h}(1-\mu^*(0))p_2 > \frac{a^h-1}{a^h}(1-\hat{\mu})\bar{p} = K^h(0,\bar{p})$ if and only if the RHS of (30) is larger than the LHS.

Replacing $(1-\mu^*(0))p_2$ with $\bar{p}(1-\hat{\mu})$, this condition on equation (30) is equivalent to

$$(\bar{p}-p_2)\frac{a^h}{a^h-1}\left(1 - e^{\frac{-a^h-1}{a^h}\frac{\bar{p}(1-\hat{\mu})}{\bar{p}-p_2}}\right) + \left(\frac{a^h-1}{a^h}p_2 - 1\right)\bar{p}(1-\hat{\mu}) > 0. \tag{31}$$

We know from the above cases that for prizes such that $\frac{\bar{p}-p_2}{p_2} = \underline{p}^h$ and $\frac{\bar{p}-p_2}{p_2} = \overline{p}^h$, that $K^h(\bar{p}-p_2, p_2) > K^h(0,\bar{p})$, and therefore the condition (31) holds. To show that the expression is positive for all prize ratios between, we show that it is concave down in p_2. The second term is linear in p_2 and will drop out of the second derivative. The derivatives of the first term are

$$\frac{\partial}{\partial p_2}(\bar{p}-p_2)\frac{a^h}{a^h-1}\left(1 - e^{\frac{-a^h-1}{a^h}\frac{\bar{p}(1-\hat{\mu})}{\bar{p}-p_2}}\right) = \frac{\bar{p}(1-\hat{\mu})}{\bar{p}-p_2}e^{\frac{-a^h-1}{a^h}\frac{\bar{p}(1-\hat{\mu})}{\bar{p}-p_2}} - \frac{a^h}{a^h-1}\left(1 - e^{\frac{-a^h-1}{a^h}\frac{\bar{p}(1-\hat{\mu})}{\bar{p}-p_2}}\right)$$

$$\begin{aligned}\frac{\partial^2}{\partial p_2^2}(\bar{p}-p_2)\frac{a^h}{a^h-1}\left(1 - e^{\frac{-a^h-1}{a^h}\frac{1-\hat{\mu}}{1-p_2}}\right) &= -\frac{\bar{p}(1-\hat{\mu})}{(\bar{p}-p_2)^2}e^{\frac{-a^h-1}{a^h}\frac{\bar{p}(1-\hat{\mu})}{\bar{p}-p_2}} + \frac{\bar{p}(1-\hat{\mu})}{(\bar{p}-p_2)^2}e^{\frac{-a^h-1}{a^h}\frac{\bar{p}(1-\hat{\mu})}{\bar{p}-p_2}} \\ &\quad - \frac{(\bar{p}(1-\hat{\mu}))^2}{(\bar{p}-p_2)^3}\frac{a^h-1}{a^h}e^{\frac{-a^h-1}{a^h}\frac{\bar{p}(1-\hat{\mu})}{\bar{p}-p_2}} < 0.\end{aligned}$$

Proof of Corollary 1

For any a^h and $\hat{\mu}$, when $p_1 = \bar{p}$ and $p_2 = 0$, $K^h(\bar{p}, 0) = \frac{a^h-1}{a^h}(1-\hat{\mu})\bar{p}$ and $R^h(\bar{p}, 0) = \bar{p}(1-\hat{\mu}/2)$. From Lemma 7, expected output is $Y(\bar{p}, 0) = \frac{\bar{p}}{2}\left(1 + (a^h-1)\hat{\mu}^2\right)$.

Given a^h, we take $\hat{\mu} < \bar{\mu}(a^h) = \frac{1}{\log(a^h)} - \frac{1}{a^h-1}$ where the equilibrium characterization is given in Appendix A.1.2. When $p_1 = p_2 = \bar{p}/2$, then from (24) and given $\hat{\mu} < \frac{a^h}{a^h-1}\left(1 - \frac{\log(a^h)}{a^h-1}\right)$, it follows that $x_{h*} > 0$ and $\mu(x_{h*}) = 0$. In this case, $K^h(\bar{p}/2, \bar{p}/2) = \frac{a^h-1}{a^h}(x_{h*} + \bar{p}/2)$.

From (25), $\bar{p}^\ell$, the price ratio cutoff between the two and three-interval case, is larger than 1 when $\hat{\mu} < \frac{a^h}{(a^h-1)^2}\left(e^{\frac{a^h-1}{a^h}} - 1\right)$, where the right hand expression is strictly positive for all $a^h > 1$. This follows because the left hand side of (25) decreases in p_1/p_2 and the right hand side increases in $\hat{\mu}$, so as $\hat{\mu}$ decreases, p_1/p_2 must increase for (25) to hold. Therefore, for $\hat{\mu} < \min\left\{\frac{a^h}{(a^h-1)^2}\left(e^{\frac{a^h-1}{a^h}} - 1\right), \frac{a^h}{a^h-1}\left(1 - \frac{\log(a^h)}{a^h-1}\right), \frac{1}{\log(a^h)} - \frac{1}{a^h-1}\right\}$ and $p_1 = p_2 = \bar{p}/2$, we are in the two interval case where $\mu^*(x_\ell^*)$ is determined by (26) and $x_{h*} = \frac{\bar{p}}{2}\left(1 + \frac{a^h-1}{a^h}(\mu^*(x_\ell^*) - \hat{\mu}) - \mu^*(x_\ell^*)\right)$. Expected output is

$$\begin{aligned} Y\left(\frac{\bar{p}}{2}, \frac{\bar{p}}{2}\right) &= \frac{\bar{p}}{2} + (a^h-1)\hat{\mu}\left(\frac{\bar{p}}{2}\left(W_1^h\left(\frac{\bar{p}}{2}, \frac{\bar{p}}{2}\right) + W_2^h\left(\frac{\bar{p}}{2}, \frac{\bar{p}}{2}\right)\right) - \frac{\bar{p}}{2} - x_{h*}\right) \\ &= \frac{\bar{p}}{2}\left(1 + (a^h-1)\hat{\mu}\left(W_1^h\left(\frac{\bar{p}}{2}, \frac{\bar{p}}{2}\right) + W_2^h\left(\frac{\bar{p}}{2}, \frac{\bar{p}}{2}\right) + \frac{\mu^*(x_\ell^*)}{a^h} + \frac{a^h-1}{a^h}\hat{\mu} - 2\right)\right). \end{aligned}$$

Therefore, to show that $Y(\bar{p}/2, \bar{p}/2) > Y(\bar{p}, 0)$ for all $\hat{\mu} < \hat{\mu}^*(a^h)$ it suffices to show that

$$W_1^h(\bar{p}/2, \bar{p}/2) + W_2^h(\bar{p}/2, \bar{p}/2) > 2 - \frac{\mu^*(x_\ell^*) - \hat{\mu}}{a^h}. \tag{32}$$

In each contest, a high ability contestant loses to another high ability contestant with 1/2 probability. The probability that a high ability contestant loses to a low ability contestant in the first contest is bounded above by $\frac{\mu^*(x_\ell^*)}{2a^h}$. In the first contest, low ability contestants choose output between x_{h*} and x_ℓ^* with probability $1 - L(x_{h*})$. This is less than $1 - F(x_{h*}) = \frac{\mu^*(x_\ell^*)}{a^h} - \frac{a^h-1}{a^h}\hat{\mu} < \frac{\mu^*(x_\ell^*)}{a^h}$. Given both high and low ability contestants choose output between x_{h*} and x_ℓ^*, the probability that the high ability contestant wins exceeds 1/2 as the belief function is increasing over this interval. In the second contest, the probability that a high ability contestant loses to a low ability contestant is bounded above by $(\mu^*(x_\ell^*))^2$. If a low ability contestant is weaker than the high ability contestant, then they will beat the high ability contestant with probability $\frac{(\mu_s-\mu_w)^2}{2a^h}$. For any outputs in the first contest, this is less than $(\mu^*(x_\ell^*))^2$.

Then, $W_1^h(\bar{p}/2, \bar{p}/2) + W_2^h(\bar{p}/2, \bar{p}/2) > 2 - 2\frac{\hat{\mu}}{2} - (1-\hat{\mu})\left(\frac{\mu^*(x_\ell^*)}{2a^h} + (\mu^*(x_\ell^*))^2\right)$ and (32) holds for any $\hat{\mu}$ such that

$$\frac{\mu^*(x_\ell^*)}{2a^h} - \frac{a^h+1}{a^h}\hat{\mu} - (\mu^*(x_\ell^*))^2 \geq 0. \tag{33}$$

From (26), the implicit derivative of $\mu^*(x_\ell^*)$ with respect to $\hat{\mu}$ is

$$\frac{\partial \mu^*(x_\ell^*)}{\partial \hat{\mu}} = \frac{1}{(\frac{a^h}{a^h-1} - 1)(e^{\frac{a^h-1}{a^h}\mu^*(x_\ell^*)} - 1)},$$

which approaches positive infinity as $\hat{\mu} \to 0$. Because $\mu^*(x_\ell^*) \to 0$ as $\hat{\mu} \to 0$, the left hand side of (33) approaches 0 as $\hat{\mu} \to 0$. The derivative of this left hand side with respect to $\hat{\mu}$ is

$$\frac{\partial \mu^*(x_\ell^*)}{\partial \hat{\mu}} \left(\frac{1}{2a^h} - 2\mu^*(x_\ell^*) \right) - \frac{a^h + 1}{a^h},$$

which approaches positive infinity as $\hat{\mu} \to 0$. Therefore, there exists a $\hat{\mu}^*(a^h)$ where (33) holds for all $\hat{\mu} < \hat{\mu}^*(a^h)$, and the result on output follows. When $Y(\bar{p}/2, \bar{p}/2) > Y(\bar{p}, 0)$, it follows from Lemma 7 and the fact that $R^h(\bar{p}/2, \bar{p}/2) < R^h(\bar{p}, 0)$ for all a^h and $\hat{\mu}$ that $K^h(\bar{p}/2, \bar{p}/2) < K^h(\bar{p}, 0)$, giving the result on payoffs.

Proof of Proposition 5

Belief distributions that arise after the first contest for different ability ratios must be equal at least at one point between 0 and 1. Let $\tilde{\mu}(\hat{\tilde{x}}) = \mu(\hat{x}) = \hat{M} \in (0, 1)$ be a point of intersection for belief distributions $\tilde{F}_\mu(M)$ and $F_\mu(M)$. As shown in the proof of Lemma 8,

$$f_\mu^*(\hat{M}) = \frac{f_1^*(\hat{x})}{\mu^{*\prime}(\hat{x})} = \frac{p_2}{p_1} \left(1 - \frac{a^h - 1}{a^h} F_1^*(\hat{x}) \right).$$

Because $\tilde{F}_1^*(\tilde{\mu^*}^{-1}(\hat{M})) = F_1^*(\mu^{*-1}(\hat{M}))$, then $\tilde{f}_\mu^*(\hat{M}) < f_\mu^*(\hat{M})$ when $\tilde{a}^h > a^h$. Given that $\tilde{f}_\mu^*(\hat{M}) < f_\mu^*(\hat{M})$, this implies that $\tilde{F}_\mu^*(M)$ crosses $F_\mu^*(M)$ exactly once from above and $\tilde{F}_\mu^*(M) <_{SOSD} F_\mu^*(M)$.

It follows that for $\tilde{a}^h > a^h$, then $\tilde{x}_{h*} \geq x_{h*}$ and $\tilde{\mu}^*(\tilde{x}_{h*}) \leq \mu^*(x_{h*})$. Then it follows directly that the equilibrium expected payoffs of the high ability contestants are higher when a^h is higher: $K^h(p_1, p_2; \tilde{a}^h) > K^h(p_1, p_2; a^h)$. Following the proof of Proposition 3, high ability players are also more likely to win the first contest: $W_1^h(p_1, p_2; \tilde{a}^h) > W_1^h(p_1, p_2; a^h)$.

Proof of Proposition 6

As shown in the proof of Lemma 8, for $M \in (0, 1)$ and $\mu^*(x) = M$, the belief distribution satisfies

$$f_\mu^*(M) = \frac{f_1^*(x)}{\mu^{*\prime}(x)} = \frac{p_2}{p_1} \left(1 - \frac{a^h - 1}{a^h} F_1^*(x) \right) = \frac{p_2}{p_1} \left(1 - \frac{a^h - 1}{a^h} F_\mu^*(M) \right).$$

Given this differential equation is not impacted by the value of $\hat{\mu}$, then for $\tilde{\hat{\mu}} > \hat{\mu}$, either $\tilde{F}_\mu^*(M)$ and $F_\mu^*(M)$ are identical, or they can never cross. This implies that if the distribution functions are not identical, then one distribution function must FOSD the other. Because

$$\int_0^1 MdF_\mu^*(M) = \hat{\mu} < \tilde{\hat{\mu}} = \int_0^1 Md\tilde{F}_\mu^*(M),$$

it follows that $\tilde{F}_\mu^*(M) >_{FOSD} F_\mu^*(M)$ when $\tilde{\hat{\mu}} > \hat{\mu}$. Given this relationship between distribution functions, it must be that $\tilde{x}_{h*} \leq x_{h*}$ and $\tilde{\mu}^*(\tilde{x}_{h*}) \geq \mu^*(x_{h*})$. Therefore payoffs of high ability contestants are lower: $K^h(p_1, p_2; \tilde{\hat{\mu}}) < K^h(p_1, p_2; \hat{\mu})$.

References

Aoyagi, Masaki (2010), "Information feedback in a dynamic tournament." *Games and Economic Behavior*, 70, 242–260.

Avery, Christopher (1998), "Strategic jump bidding in english auctions." *The Review of Economic Studies*, 65, 185–210.

Barbieri, Stefano and Marco Serena (2018), "Biasing unbiased dynamic contests." *Working Paper*.

Baye, Michael R, Dan Kovenock, and Casper G De Vries (1993), "Rigging the lobbying process: an application of the all-pay auction." *The American Economic Review*, 83, 289–294.

Bergemann, Dirk and Johannes Hörner (2018), "Should first-price auctions be transparent?" *American Economic Journal: Microeconomics*, 10, 177–218.

Bonatti, Alessandro, Gonzalo Cisternas, and Juuso Toikka (2017), "Dynamic oligopoly with incomplete information." *The Review of Economic Studies*, 84, 503–546.

Breig, Zachary and Greg Kubitz (2021), "Biased contest judges." *Working Paper*.

Che, Yeon-Koo and Ian Gale (2003), "Optimal design of research contests." *American Economic Review*, 93, 646–671.

Chen, Jian, Zhonghong Kuang, and Jie Zheng (2017), "Bayesian persuasion in sequential tullock contests." Technical report.

Chen, Zhuoqiong (2021), "Optimal information exchange in contests." *Journal of Mathematical Economics*, 102518.

Crawford, Vincent P and Joel Sobel (1982), "Strategic information transmission." *Econometrica: Journal of the Econometric Society*, 1431–1451.

Denter, P, John Morgan, and Dana Sisak (2021), "Showing off or laying low? the economics of psych-outs." *American Economic Journal: Microeconomics*.

Ederer, Florian (2010), "Feedback and motivation in dynamic tournaments." *Journal of Economics & Management Strategy*, 19, 733–769.

Ford, Weixing, Zeng Lian, Jaimie W Lien, and Jie Zheng (2020), "Information sharing in a contest game with group identity." *Economics Letters*, 189, 109000.

Harris, Christopher and John Vickers (1987), "Racing with uncertainty." *The Review of Economic Studies*, 54, 1–21.

Heijnen, Pim and Lambert Schoonbeek (2017), "Signaling in a rent-seeking contest with one-sided asymmetric information." *Journal of Public Economic Theory*, 19, 548–564.

Hinnosaar, Toomas (2021), "Optimal sequential contests." *Working Paper*.

Hörner, Johannes and Nicolas Sahuguet (2007), “Costly signalling in auctions.” *The Review of Economic Studies*, 74, 173–206.

Kamenica, Emir and Matthew Gentzkow (2011), “Bayesian persuasion.” *American Economic Review*, 101, 2590–2615.

Klein, Arnd Heinrich and Armin Schmutzler (2017), “Optimal effort incentives in dynamic tournaments.” *Games and Economic Behavior*, 103, 199–224.

Konrad, Kai A (2012), “Dynamic contests and the discouragement effect.” *Revue d’économie politique*, 122, 233–256.

Konrad, Kai A and Dan Kovenock (2009), “Multi-battle contests.” *Games and Economic Behavior*, 66, 256–274.

Konrad, Kai A and Florian Morath (2018), “To deter or to moderate? alliance formation in contests with incomplete information.” *Economic Inquiry*, 56, 1447–1463.

Kovenock, Dan, Florian Morath, and Johannes Münster (2015), “Information sharing in contests.” *Journal of Economics & Management Strategy*, 24, 570–596.

Lu, Jingfeng, Hongkun Ma, and Zhe Wang (2018), “Ranking disclosure policies in all-pay auctions.” *Economic Inquiry*, 56, 1464–1485.

Mirman, Leonard J, Larry Samuelson, and Amparo Urbano (1993), “Duopoly signal jamming.” *Economic Theory*, 3, 129–149.

Münster, Johannes (2009), “Repeated contests with asymmetric information.” *Journal of Public Economic Theory*, 11, 89–118.

Ortega Reichert, Armando (2000), “A sequential game with information flow.” In *The Economic Theory of Auctions I*, 232–54, Edward Elgar Publishing Limited.

Ridlon, Robert and Jiwoong Shin (2013), “Favoring the winner or loser in repeated contests.” *Marketing Science*, 32, 768–785.

Rosen, Sherwin (1986), “Prizes and incentives in elimination tournaments.” *The American Economic Review*, 701–715.

Serena, Marco (2021), “Harnessing beliefs to optimally disclose contestants’ types.” *Economic Theory*, 1–30.

Siegel, Ron (2010), “Asymmetric contests with conditional investments.” *The American Economic Review*, 100, 2230–2260.

Siegel, Ron (2014), “Asymmetric all-pay auctions with interdependent valuations.” *Journal of Economic Theory*, 153, 684–702.

Spence, Michael (1973), “Job market signaling.” *The Quarterly Journal of Economics*, 87, 355–374.

Terwiesch, Christian and Yi Xu (2008), “Innovation contests, open innovation, and multiagent problem solving.” *Management Science*, 54, 1529–1543.

Wu, Zenan and Jie Zheng (2017), "Information sharing in private value lottery contest." *Economics Letters*, 157, 36–40.

Zhang, Jun and Ruqu Wang (2009), "The role of information revelation in elimination contests." *The Economic Journal*, 119, 613–641.

Zhang, Jun and Junjie Zhou (2016), "Information disclosure in contests: A bayesian persuasion approach." *The Economic Journal*, 126, 2197–2217.

Zheng, J, J Chen, and Z Kuang (2018), "Bayesian persuasion in all-pay auction contests." *Working Paper*.

C Online Appendix

C.1 Uniqueness of equilibrium with convex cost

In this section we show that Theorem 1 holds for more general cost of effort. We denote the cost function of effort as $c(e)$, which is the same for high and low ability contestants. The cost function is assumed to be twice differentiable on the non-negative reals, strictly increasing and weakly convex, with the cost of zero effort being zero.

Lemma C.1. *In any equilibrium and for any history η_2, $BR^{\ell}_{i2}(\sigma^*, \eta_2) \cup BR^{h}_{i2}(\sigma^*, \eta_2) = [0, x^*]$ for $i = s, w$ and $x^* > 0$. Equilibrium output distributions, $H^*_{2s}(x|\eta_2), L^*_{2s}(x|\eta_2)$, $H^*_{2w}(x|\eta_2)$, and $L^*_{2w}(x|\eta_2)$, are continuous on $(0, x^*]$ and for $x \in BR^{\ell}_{i2}(\sigma^*, \eta_2)$ and $x' \in BR^{h}_{i2}(\sigma^*, \eta_2)$, then $x \leq x'$.*

Proof. The proof follows in three steps:

(1) We first show there is no output at which both contestants have an atom and if a contestant has an atom, it is at zero. It follows that $F^*_{i2}(x|\eta_2)$ are continuous on $(0, \infty)$ for $i = s, w$ and any η_2.

Assume both contestants produce x with positive probability. Because the cost of effort is continuous, either contestant can improve payoffs by producing output slightly above this atom. Then x is not a best response of that contestant, a contradiction.

Assume that contestant i produces $x > 0$ with positive probability. Then by the continuity of the cost function in output, there is a $\delta > 0$ such that for all $\hat{x} \in (x - \delta, x)$, $\hat{x} \notin BR^{\theta}_{-i2}(\sigma^*, \eta_2)$ for $\theta = \ell, h$. This implies, that contestant i would do better by playing $x - \delta/2$, and therefore $x \notin BR^{\theta}_{i2}(\sigma^*, \eta_2)$, a contradiction.

(2) Next, if $\hat{x} > 0$ is not a best response for any ability of one of the contestants, then for all $x > \hat{x}$, x is not a best response for either type of either contestant.

Step (1) implies that equilibrium payoffs are continuous over positive outputs. Given $x' \notin BR^{\ell}_{i2}(\sigma^*, \eta_2) \cup BR^{h}_{i2}(\sigma^*, \eta_2)$, for some $i = s, w$, $\exists \tilde{x}^h, \tilde{x}^{\ell}$ for which $\mathbb{E}[\pi_{i2}(\tilde{x}^h)|a^h, \sigma^*, \eta_2] > \mathbb{E}[\pi_{i2}(x')|a^h, \sigma^*, \eta_2] + \varepsilon$ and $\mathbb{E}[\pi_{i2}(\tilde{x}^{\ell})|a^{\ell}, \sigma^*, \eta_2] > \mathbb{E}[\pi_{i2}(x')|a^{\ell}, \sigma^*, \eta_2] + \varepsilon$. Then, every output in the neighborhood of x' in this neighborhood cannot be a best response of either type of contestant i, and therefore also cannot be a best response for any type of contestant $-i$, who could improve expected payoffs by lowering output.

Define $X^* = \{x | x > \hat{x}$ and $x \in BR^{\ell}_{i2}(\sigma^*, \eta_2) \cup BR^{h}_{i2}(\sigma^*, \eta_2)\}$. Let $x_* = \inf\{X^*\}$. Then, there is a neighborhood below x_* for which all outputs are not best responses for any ability type of either contestant. Bt continuity of payoffs, this would imply that there is an $x \in X^*$ that gives lower expected payoffs than $\hat{x}$ for both types of each contestant, a contradiction. Therefore, x_* does not exist and X^* is empty. This implies that $\sup\{BR^{\ell}_{s2}(\sigma^*, \eta_2) \cup BR^{h}_{s2}(\sigma^*, \eta_2)\} = \sup\{BR^{\ell}_{w2}(\sigma^*, \eta_2) \cup BR^{h}_{w2}(\sigma^*, \eta_2)\} \equiv x^*$ and the combined best response sets of each contestant is $[0, x^*]$.

(3) In any equilibrium, for $x \in BR^{\ell}_{i2}(\sigma^*, \eta_2)$ and $x' \in BR^{h}_{i2}(\sigma^*, \eta_2)$, then $x \leq x'$.

Assume otherwise, $\exists x \in BR^{\ell}_{i2}(\sigma^*, \eta_2)$ and $x' \in BR^{h}_{i2}(\sigma^*, \eta_2)$, with $x > x'$. Then

$$\mathbb{E}[\pi_{i2}(x) - \pi_{i2}(x')|a^{\ell}, \sigma^*, \eta_2] = p_2(F^*_{i2}(x|\eta_2) - F^*_{i2}(x'|\eta_2)) - (c(x) - c(x')) \geq 0.$$

Because the cost function is increasing and weakly convex, and $x > x'$, then

$$\mathbb{E}[\pi_{i2}(x) - \pi_{i2}(x')|a^h, \sigma^*, \eta_2] = p_2(F^*_{i2}(x|\eta_2) - F^*_{i2}(x'|\eta_2)) - (c(x/a^h) - c(x'/a^h)) > 0.$$

This contradicts $x' \in BR^h_{i2}(\sigma^*, \eta_2)$.

□

Proof of Proposition 7

From Lemma C.1, there are x^*_s and x^*_w such that $x^*_i = \sup\{BR^\ell_{i2}(\sigma^*, \eta_2)\} = \inf\{BR^h_{i2}(\sigma^*, \eta_2)\}$, for $i = s, w$, and $x^* \equiv \sup\{BR^h_{i2}(\sigma^*, \eta_2)\}$ which is the same for $i = s, w$.

Each contestant must be indifferent between all $x \in (x^*_i, x^*)$ when they have high ability. Each high ability contestant has the same marginal cost of output so indifference implies that the expected output density must also be the same: $f^*_{s2}(x) = f^*_{w2}(x)$ for $x \in (\max\{x^*_s, x^*_w\}, x^*)$. Since $f^*_{i2}(x) = \mu_i h^*_{i2}(x)$ for all $x \in (x^*_i, x^*)$, then $h^*_{s2}(x) \leq h^*_{w2}(x)$ for all $x \in (\max\{x^*_s, x^*_w\}, x^*)$. Then $H_{i2}(x^*_i) = 0$, requires that $x^*_s \leq x^*_w$.

Therefore, for the remainder of the construction, there are three intervals to consider: the best response set of the low types of both the stronger and the weaker contestants, $0 \leq x \leq x^*_s$, the best response set of the low type of the weaker contestant and the high type of the strong contestant, $x^*_s \leq x \leq x^*_w$, and best response set of the high types of each contestant, $x^*_w \leq x \leq x^*$.

Within their best response sets, contestants must be indifferent between all output levels. For example, the strong contestant with high ability must be indifferent to picking all outputs between x^*_s and x^*. This puts a condition on $F_{w2}(x)$, the output distribution of the weak contestant, on the interval $[x^*_s, x^*]$:

$$p_2 F^*_{w2}(x') - c\left(\frac{x}{a^h}\right) = p_2 F^*_{w2}(x') - c\left(\frac{x'}{a^h}\right).$$

Rearranging and taking the limit as $x \to x'$, $\lim_{x \to x'} \frac{F^*_{w2}(x) - F^*_{w2}(x')}{c(\frac{x}{a^h}) - c(\frac{x'}{a^h})} = \frac{1}{p_2}$. Then the output density of the strong contestant is

$$f^*_{w2}(x') = \lim_{x \to x'} \frac{F^*_{w2}(x) - F^*_{w2}(x')}{c\left(\frac{x}{a^h}\right) - c\left(\frac{x'}{a^h}\right)} \frac{c\left(\frac{x}{a^h}\right) - c\left(\frac{x'}{a^h}\right)}{a^h\left(\frac{1}{a^h}(x - x')\right)} = \frac{c'\left(\frac{x'}{a^h}\right)}{p_2 a^h}.$$

A similar calculation on each interval for each contestant allows us to characterize the densities of the output on each of the intervals below.

- $x^*_w \leq x \leq x^*$: $h^*_{s2}(x) = \frac{c'(x/a^h)}{p_2 a^h \mu_s}$, $h^*_{w2}(x) = \frac{c'(x/a^h)}{p_2 a^h \mu_w}$, $f^*_{s2}(x) = f^*_{w2}(x) = \frac{c'(x/a^h)}{p_2 a^h}$.
- $x^*_s \leq x \leq x^*_w$: $h^*_{s2}(x) = \frac{c'(x)}{p_2 \mu_s}$, $\ell^*_{w2}(x) = \frac{c'(x/a^h)}{p_2 a^h (1-\mu_w)}$, $f^*_{s2}(x) = \frac{c'(x)}{p_2}$, $f^*_{w2}(x) = \frac{c'(x/a^h)}{p_2 a^h}$.
- $0 \leq x \leq x^*_s$: $\ell^*_{s2}(x) = \frac{c'(x)}{p_2(1-\mu_s)}$, $\ell^*_{w2}(x) = \frac{c'(x)}{p_2(1-\mu_w)}$, $f^*_{s2}(x) = f^*_{w2}(x) = \frac{c'(x)}{p_2}$.

It remains to characterize the cutoff points, x^*_w, x^*_s and x^*, and $L^*_{w2}(0)$. In equilibrium, the distribution of output for each contestant must satisfy

$$L^*_{i2}(x^*_i) = 1, \quad H^*_{i2}(x^*_i) = 0, \quad F^*_{i2}(x^*_i) = 1 - \mu_i, \text{ and } \quad F^*_{i2}(x^*) = 1.$$

Additionally, the strong contestant chooses no effort with zero probability, so $L^*_{s2}(0) = 0$. Using $L^*_{s2}(x^*_s) = 1$ and the definition of $\ell^*_{s2}(x)$ on $[0, x^*_s]$, we calculate x^*_s.

$$\int_0^{x^*_s} \ell^*_{s2}(x)dx = L^*_{s2}(x^*_s) - L^*_{s2}(0) = \frac{c(x^*_s)}{p_2(1-\mu_s)} = 1$$

Then $c(x^*_s) = p_2(1-\mu_s)$, so that $x^*_s = c^{-1}(p_2(1-\mu_s))$. Similarly, $x^*_w = c^{-1}(p_2(1-\mu_w))$. From these endpoints we can calculate x^*.

$$\int_{x^*_s}^{x^*_w} h^*_{s2}(x)dx = \frac{c(x^*_w) - c(x^*_s)}{\mu_s} = \frac{(1-\mu_w) - (1-\mu_s)}{\mu_s} = \frac{\mu_s - \mu_w}{\mu_s}$$

$$\int_{x^*_w}^{x^*} h^*_{s2}(x_s)dx = 1 - \frac{\mu_s - \mu_w}{\mu_s} = \frac{\mu_w}{\mu_s}$$

$$\int_{x^*_w}^{x^*} f^*_{s2}(x_s)dx = \frac{1}{p_2}\left(c\left(\frac{x^*}{a^h}\right) - c\left(\frac{c^{-1}(p_2(1-\mu_w))}{a^h}\right)\right) = \mu_w$$

$$x^* = a^h c^{-1}\left(p_2\mu_w + c\left(\frac{c^{-1}(p_2(1-\mu_w))}{a^h}\right)\right)$$

Lastly, we pin down the probability that the weaker contestant exerts no effort.

$$\int_{x^*_s}^{x^*_w} \ell^*_{w2}(x)dx = \frac{1}{p_2(1-\mu_w)}\left[c\left(\frac{c^{-1}(p_2(1-\mu_w))}{a^h}\right) - c\left(\frac{c^{-1}(p_2(1-\mu_s))}{a^h}\right)\right]$$

$$\int_0^{x^*_s} \ell^*_{w2}(x)dx = \frac{c(c^{-1}(p_2(1-\mu_s)))}{p_2(1-\mu_w)} - 0 = \frac{1-\mu_s}{1-\mu_w}$$

$$L^*_{w2}(0) = \mu_s - \mu_w - \frac{1}{p_2(1-\mu_w)}\left[c\left(\frac{c^{-1}(p_2(1-\mu_w))}{a^h}\right) - c\left(\frac{c^{-1}(p_2(1-\mu_s))}{a^h}\right)\right]$$

Lemma C.2. *Given history $\eta_2 = (x_{s1}, x_{w1})$ with associated beliefs, $\mu_s \geq \mu_w$, the second contest continuation value of each contestant conditional on their ability are*

$$\begin{aligned}
v^h_s(\mu_s, \mu_w) &= v^h_w(\mu_w, \mu_s) = p_2(1-\mu_w) - c\left(\frac{c^{-1}(p_2(1-\mu_w))}{a^h}\right), \\
v^\ell_s(\mu_s, \mu_w) &= p_2(\mu_s - \mu_w) - \left[c\left(\frac{c^{-1}(p_2(1-\mu_w))}{a^h}\right) - c\left(\frac{c^{-1}(p_2(1-\mu_s))}{a^h}\right)\right], \\
&= v^\ell_w(\mu_w, \mu_s) = 0.
\end{aligned}$$

Proof. The expected payoffs of a high ability contestant are equal to the value of winning less the cost of producing output x^*, as producing x^* guarantees a win.

$$v_s^h(\mu_s, \mu_w) = v_w^h(\mu_w, \mu_s) = p_2 - c(x^*/a^h) = p_2(1-\mu_w) - c\left(\frac{c^{-1}(p_2(1-\mu_w))}{a^h}\right)$$

The expected payoffs of low ability contestants is equal to the probability they win given they exert no effort. This is the probability the other contestant puts in no effort.[23]

$$\begin{aligned} v_s^\ell(\mu_s, \mu_w) &= p_2(1-\mu_w)L_{w2}^*(0) \\ &= p_2(\mu_s - \mu_w) - \left[c\left(\frac{c^{-1}(p_2(1-\mu_w))}{a^h}\right) - c\left(\frac{c^{-1}(p_2(1-\mu_s))}{a^h}\right)\right] \\ v_w^\ell(\mu_w, \mu_s) &= p_2(1-\mu_s)L_{s2}^*(0) = 0 \end{aligned}$$

□

Proposition C.1. *Let $F_{\mu_{-i}}(M) = \Pr(\mu_{-i} \leq M)$ be the belief distribution of contestant $-i$'s ability resulting from the first contest, and let $\underline{M} = \sup\{M | F_{\mu_{-i}}(M) = 0\}$ and $\overline{M} = \inf\{M | F_{\mu_{-i}}(M) = 1\}$. For all $\mu_i \in (\underline{M}, \overline{M})$, expected payoffs in the second contest decrease for high ability players as μ_i increases, $\frac{\partial}{\partial \mu_i}\mathbb{E}[v_i^h(\mu_i, \mu_{-i})] < 0$, and increase with μ_i for low ability players, $\frac{\partial}{\partial \mu_i}\mathbb{E}[v_i^\ell(\mu_i, \mu_{-i})] > 0$.*

Proof. In the second contest, for a given pair of beliefs, contestants will expect the following payoffs:

$$v_i^h(\mu_i, \mu_{-i}) = p_2(1 - \min\{\mu_i, \mu_{-i}\}) - c\left(\frac{c^{-1}(p_2(1-\min\{\mu_i, \mu_{-i}\}))}{a^h}\right)$$

$$v_i^\ell(\mu_i, \mu_{-i}) = \begin{cases} p_2(\mu_i - \mu_{-i}) - \left[c\left(\frac{c^{-1}(p_2(1-\mu_{-i}))}{a^h}\right) - c\left(\frac{c^{-1}(p_2(1-\mu_i))}{a^h}\right)\right], & \text{if } \mu_i \geq \mu_{-1} \\ 0, & \text{otherwise} \end{cases}.$$

For a high ability contestant believed to be high ability with probability μ_i and with opponent's belief distribution, $F_{\mu_{-i}}$, the expected payoff in the second contest is

$$\begin{aligned} &\mathbb{E}[v_i^h(\mu_i, \mu_{-i})] \\ &= \int_0^1 \left(p_2(1-\min\{\mu_i, \mu_{-i}\}) - c\left(\frac{c^{-1}(p_2(1-\min\{\mu_i, \mu_{-i}\}))}{a^h}\right)\right) dF_{\mu_{-i}}(\mu_{-i}). \end{aligned}$$

As the opponent believes the contestant is stronger, the change in expected payoff is

$$\frac{\partial}{\partial \mu_i}\mathbb{E}_{\mu_{-i}}[v_i^h(\mu_i, \mu_{-i})] = \left(p_2 + \frac{\partial}{\partial \mu_i} c\left(\frac{c^{-1}(p_2(1-\mu_i))}{a^h}\right)\right)(F_{\mu_{-i}}(\mu_i) - 1).$$

[23] Here we technically are assuming the contestant wins all ties at zero, but if the agent exerts a tiny amount of effort and we let that effort shrink to zero, then this is the payoff of the agent in the limit. Since the payoffs are continuous, these limits must be the payoffs of the low ability contestants.

For a low ability contestant, the expected payoff is

$$\mathbb{E}[v_i^\ell(\mu_i,\mu_{-i})] = \int_0^{\mu_i} \left(p_2(\mu_i-\mu_{-i}) + c\left(\frac{c^{-1}(p_2(1-\mu_i))}{a^h}\right) - c\left(\frac{c^{-1}(p_2(1-\mu_{-i}))}{a^h}\right)\right) dF_{\mu_{-i}}(\mu_{-i}),$$

with change in expected payoff

$$\frac{\partial}{\partial \mu_i}\mathbb{E}[v_i^\ell(\mu_i,\mu_{-i})] = \left(p_2 + \frac{\partial}{\partial \mu_i} c\left(\frac{c^{-1}(p_2(1-\mu_i))}{a^h}\right)\right) F_{\mu_{-i}}(\mu_i).$$

Given the assumptions on the cost of effort, $c'(e)>0$ and $c''(e)\geq 0$,

$$\frac{\partial}{\partial \mu_i} c\left(\frac{c^{-1}(p_2(1-\mu_i))}{a^h}\right) = -\frac{1}{a^h}c'\left(\frac{c^{-1}(p_2(1-\mu_i))}{a^h}\right)\frac{1}{c'(c^{-1}(p_2(1-\mu_i)))} \in \left[-\frac{p_2}{a^h},0\right).$$

Define $d(\mu_i) \equiv \left[p_2 + \frac{\partial}{\partial\mu_i} c\left(\frac{c^{-1}(p_2(1-\mu_i))}{a_h}\right)\right]$. For all μ_i, $d(\mu_i) \in \left[\frac{p_2(a^h-1)}{a^h}, p_2\right)$. It follows that

$$\frac{\partial}{\partial \mu_i} E[v_i^h(\mu_i,\mu_{-i})] = d(\mu_i)(F_{\mu_{-i}}(\mu_i)-1) \text{ and } \frac{\partial}{\partial \mu_i} E[v_i^\ell(\mu_i,\mu_{-i})] = d(\mu_i)F_{\mu_{-i}}(\mu_i),$$

where the former derivative is strictly negative and the later is strictly positive when $\mu_i \in (\underline{M}_{-i}, \overline{M}_{-i})$. □

Lemma C.3. *In every SPBE, $\mu^*(x)$ is weakly increasing in x for all $x \in X_{i1} \equiv X_{i1}^h \cup X_{i1}^\ell$.*

Proof. Let $x, x' \in X_{i1}$ such that $x < x'$ and $\mu(x) > \mu(x')$. Then $0 \leq \mu(x') < \mu(x) \leq 1$ which implies $x \in X_{i1}^h \subseteq BR_{i1}^h$ and $x' \in X_{i1}^\ell \subseteq BR_{i1}^\ell$. Best responses require

$$\begin{aligned} p_1\mathbb{E}[w_i(x',x_{-i1})] - c(x') + \mathbb{E}[v_i^\ell(\mu(x'),\mu(x_{-i1}))] \\ \geq p_1\mathbb{E}[w_i(x,x_{-i1})] - c(x) + \mathbb{E}[v_i^\ell(\mu(x),\mu(x_{-i1}))], \text{ and} \\ p_1\mathbb{E}[w_i(x',x_{-i1})] - c(x'/a^h) + \mathbb{E}[v_i^h(\mu(x'),\mu(x_{-i1}))] \\ \leq p_1\mathbb{E}[w_i(x,x_{-i1})] - c(x/a^h) + \mathbb{E}[v_i^h(\mu(x),\mu(x_{-i1}))]. \end{aligned}$$

This implies that

$$\begin{aligned} p_1(\mathbb{E}[w_i(x',x_{-i1})] - \mathbb{E}[w_i(x,x_{-i1})]) + \mathbb{E}[v_i^\ell(\mu(x'),\mu(x_{-i1}))] - \mathbb{E}[v_i^\ell(\mu(x),\mu(x_{-i1}))] \\ \geq c(x') - c(x), \text{ and} \\ p_1(\mathbb{E}[w_i(x',x_{-i1})] - \mathbb{E}[w_i(x,x_{-i1})]) + \mathbb{E}[v_i^h(\mu(x'),\mu(x_{-i1}))] - \mathbb{E}[v_i^h(\mu(x),\mu(x_{-i1}))] \\ \leq c(x'/a^h) - c(x/a^h). \end{aligned}$$

From Proposition C.1, $\mu(x) > \mu(x')$ implies

$$\mathbb{E}[v_i^\ell(\mu(x'), \mu(x_{-i1}))] - \mathbb{E}[v_i^\ell(\mu(x), \mu(x_{-i1}))] \leq 0,$$
$$\text{and } \mathbb{E}[v_i^h(\mu(x'), \mu(x_{-i1}))] - \mathbb{E}[v_i^h(\mu(x), \mu(x_{-i1}))] \geq 0.$$

Combining the previous inequalities,

$$c(x') - c(x) \leq p_1(\mathbb{E}[w_i(x', x_{-i1})] - \mathbb{E}[w_i(x, x_{-i1})]) \leq c(x'/a^h) - c(x/a^h),$$

which cannot be true given $a^h > 1$, $c''(x) \geq 0$ and $c'(x) > 0$. □

Proposition C.2. *Given $p_1 = 0$ and $p_2 > 0$, there is a unique SPBE where $X_{i1} = \{0\}$.*

Proof. Equilibrium conditions are satisfied when $H_1^*(x) = L_1^*(x) = 1$ for $x \geq 0$ and 0 otherwise (i.e. the output densities of both high and low ability contestants consist of a single mass point at $x = 0$), $\mu^*(x) = \hat{\mu}$ for $x \geq 0$, and second period distribution functions are as characterized in (5).

To show that there can be no equilibrium where $\tilde{x} \in X_{i1}$, such that $\tilde{x} > 0$, assume that there is. Then $\tilde{x} \in BR_{i1}^\ell(\sigma_{-i}) \cup BR_{i1}^h(\sigma_{-i})$. If $\tilde{x} \in BR_{i1}^\ell(\sigma_{-i})$ then $\mathbb{E}[v_i^\ell(\mu(\tilde{x}), \mu(x_{-i1}))] - \mathbb{E}[v_i^\ell(\mu(0), \mu(x_{-i1}))] \geq c(\tilde{x}) > 0$ which implies that $\mu(\tilde{x}) > \mu(0) \geq 0$. Because $\mu(\tilde{x}) > 0$, equilibrium conditions on the belief function require that $\tilde{x} \in X_{i1}^h \subset BR_{i1}^h$ and therefore $\mathbb{E}[v_i^h(\mu(\tilde{x}), \mu(x_{-i1}))] - \mathbb{E}[v_i^h(\mu(0), \mu(x_{-i1}))] \geq c\left(\frac{\tilde{x}}{a^h}\right) > 0$, which cannot be true when $\mu(\tilde{x}) > \mu(0)$, a contradiction.

If $\tilde{x} \in BR_{i1}^h(\sigma_{-i}) \setminus BR_{i1}^\ell(\sigma_{-i})$, then $\mu(\tilde{x}) < \mu(0)$ which implies by Lemma C.3 that $0 \notin X_{i1}$. This however would require the existence of a positive output in $BR_{i1}^\ell(\sigma_{-i})$, which we just ruled out. □

Lemma C.4. *Let $p_1 > 0$. For any SPBE, first contest output distributions are continuous and therefore $\mathbb{E}[w_i(x_{i1}, x_{-i1})|x_{i1}] = F_1^*(x_{i1}) = \hat{\mu}H_1^*(x_{i1}) + (1 - \hat{\mu})L_1^*(x_{i1})$ is continuous.*

Proof. In a symmetric equilibrium, if an output is played with positive probability by one type of contestant, then it must be played with positive probability by both contestants of this type. Let $\tilde{x} \in \{X_{i1}^\ell \cup X_{i1}^h\}$ be played with probability $q > 0$. Then

$$\mathbb{E}[w_i(\tilde{x}, x_{-i1})] + \frac{q}{2} \leq \mathbb{E}[w_i(x, x_{-i1})] \text{ for all } x > \tilde{x}.$$

Since, $\tilde{x} \in BR_{i1}^\theta(\sigma_{-i})$, for some θ, then for all $x \geq 0$,

$$\begin{aligned} p_1\mathbb{E}[w_i(\tilde{x}, x_{-i1})] - c(\tilde{x}/a^\theta) + \mathbb{E}[v_i^\theta(\mu_i(\tilde{x}), \mu_{-i})] \\ \geq p_1\mathbb{E}[w_i(x, x_{-i1})] - c(x/a^\theta) + \mathbb{E}[v_i^\theta(\mu_i(x), \mu_{-i})]. \end{aligned}$$

Combining the above inequalities,

$$p_1\frac{q}{2} \leq \mathbb{E}[v_i^\theta(\mu_i(\tilde{x}), \mu_{-i})] - \mathbb{E}[v_i^\theta(\mu_i(x), \mu_{-i})] + c(x/a^\theta) - c(\tilde{x}/a^\theta).$$

By continuity of the cost function, $\exists\varepsilon > 0$ such that for all $x \in (\tilde{x}, \tilde{x} + \varepsilon)$, we have

$c\left(\frac{\tilde{x}+\varepsilon}{a^\theta}\right) - c\left(\frac{\tilde{x}}{a^\theta}\right) < p_1\frac{q}{2}$. Then for each x in this range

$$\mathbb{E}[v_i^\theta(\mu_i(\tilde{x}), \mu_{-i})] - \mathbb{E}[v_i^\theta(\mu_i(x), \mu_{-i})] > 0. \tag{34}$$

From Proposition 1, if $\theta = \ell$, then $\mu_i(\tilde{x}) > \mu_i(x)$ and $\tilde{x} \in \{X_{i1}^\ell \cap X_{i1}^h\}$. Similarly, if $\theta = h$, then $\mu_i(\tilde{x}) < \mu_i(x)$ and $\tilde{x} \in \{X_{i1}^\ell \cap X_{i1}^h\}$. In either case, $\tilde{x} \in \{BR_{i1}^\ell(\sigma_{-i}) \cap BR_{i1}^h(\sigma_{-i})\}$. However, (34) cannot hold for both $\theta = \ell$ and $\theta = h$, a contradiction. □

Lemma C.5. *Let $p_1, p_2 > 0$. Define $x_{\ell *} = \inf X_{i1}^\ell$, $x_\ell^* = \sup X_{i1}^\ell$, $x_{h*} = \inf X_{i1}^h$, and $x_h^* = \sup X_{i1}^h$. In any SPBE, the best response sets of low and high ability contestants in the first contest are intervals with $BR_{i1}^\ell(\sigma^*) = [0, x_\ell^*]$, $BR_{i1}^h(\sigma^*) = [x_{h*}, x_h^*]$ and $0 = x_{\ell *} \le x_{h*} < x_\ell^* \le x_h^*$.*

Proof. From Lemma C.4 we now can use the fact that $L_1^*(x)$ and $H_1^*(x)$, and therefore $F_1^*(x)$, are continuous in x and we have that in equilibrium $\mathbb{E}[w_i(x, x_{-i1})] = \Pr(x_{-i1} < x|\sigma_{-i}^*) = \Pr(x_{-i1} \le x|\sigma_{-i}^*) = F_1^*(x)$. Combined with Lemma C.3, we have $\Pr(\mu^*(x_{-i1}) < \mu^*(x)|\sigma_{-i}^*) \le \mathbb{E}[w_i(x, x_{-i1})] = F_1^*(x) \le \Pr(\mu^*(x_{-i1}) \le \mu^*(x)|\sigma_{-i}^*) = F_{\mu_{-i}}(\mu^*(x))$. The proof follows in four steps.

(1) We first show that $x_{\ell *} = 0$. We do this by first showing that $x_{\ell *} \le x_{h*}$, and then showing that $x_{\ell *}$ cannot be larger than zero.

Let $x_{h*} < x_{\ell *}$. Since $x_{h*} = \inf X_{i1}^h$, $\forall \varepsilon > 0$, $\exists x_\varepsilon$ such that $x_{h*} \le x_\varepsilon < x_{h*} + \varepsilon$ and $x_\varepsilon \in X_{i1}^h$. In particular, this holds for $\varepsilon^* = x_{\ell *} - x_{h*}$. Then $x_{\varepsilon^*} \in \{X_{i1}^h \setminus X_{i1}^\ell\}$ and $\mu^*(x_{\varepsilon^*}) = 1$. However, from Lemma C.3 we would have $\mu^*(x) = 1$ for all $x \in X_{i1}^\ell$, which cannot hold. Therefore $x_{h*} \ge x_{\ell *}$.

If $0 < x_{\ell *} < x_{h*}$, then by Lemma C.4, $\exists \delta$ with $0 < \delta < x_{h*} - x_{\ell *}$ such that $\forall x \in (x_{\ell *}, x_{\ell *} + \delta)$ we have $|p_1(F_1^*(x) - F_1^*(0))| = |p_1(F_1^*(x) - F_1^*(x_{\ell *}))| < c(x_{\ell *})$. Let $x_\delta \in X_{i1}^\ell \cap (x_{\ell *}, x_{\ell *} + \delta)$. Then $\mu(x_\delta) = 0$ and $p_1(F_1^*(x_\delta) - F_1^*(0)) < c(x_\delta)$. However this implies

$$p_1 F_1^*(0) + \mathbb{E}[v_i^\ell(\mu(0), \mu_{-i})] > p_1 F_1^*(x_\delta) + \mathbb{E}[v_i^\ell(\mu(x_\delta), \mu_{-i})] - c(x_\delta),$$

and therefore $x_\delta \notin BR_{i1}^\ell(\sigma_{-i})$, a contradiction.

If $0 < x_{\ell *} = x_{h*}$, then $\exists x_\ell, x_h$ such that $x_\ell \le x_h$, $x_\ell \in X_{i1}^\ell$, $x_h \in X_{i1}^h$, and $p_1(F_1^*(x_\ell) - F_1^*(x_{\ell *})) = p_1 F_1^*(x_\ell) < c(x_{\ell *}) < c(x_\ell)$ and $p_1(F_1^*(x_h) - F_1^*(x_{h*})) = p_1 F_1^*(x_h) < c(x_{h*}/a_h) < c(x_h/a_h)$, by the continuity of $F_1^*(x)$. It follows that $x_\ell \in X_{i1}^\ell$ implies

$$p_1 F_1^*(x_\ell) - c(x_\ell) + E[v_i^\ell(\mu(x_\ell), \mu_{-i})] \ge p_1 F_1^*(0) - c(0) + \mathbb{E}[v_i^\ell(\mu(0), \mu_{-i})]$$

and $\mathbb{E}[v_i^\ell(\mu(x_\ell), \mu_{-i})] > \mathbb{E}[v_i^\ell(\mu(0), \mu_{-i})]$, which requires $\mu(x_\ell) > \mu(0)$.

Similarly, $x_h \in X_{i1}^h$ implies that $\mu(x_h) < \mu(0)$. Combining these two inequalities leads to $\mu(x_h) < \mu(x_\ell)$. This contradicts Lemma C.3. Therefore we must have $0 = x_{\ell *} \le x_{h*}$.

(2) We next show that $x_{h*} \le x_\ell^*$.

If $x_\ell^* < x_{h*}$, then $\forall x \in (x_\ell^*, x_{h*})$, $x \notin \{X_{i1}^\ell \cup X_{i1}^h\}$. Let $\tilde{x} = \frac{x_\ell^* + x_{h*}}{2}$ and $\varepsilon = c(x_{h*}/a^h) - c(\tilde{x}/a^h)$. There is a $\delta > 0$ such that $\forall x \in (x_{h*}, x_{h*} + \delta)$, $p_1(F_1^*(x) - F_1^*(x_{h*})) < \varepsilon$. Pick an

x_δ such that $x_\delta \in (x_{h*}, x_{h*}+\delta)$ and $x_\delta \in X_{i1}^h$. Then $p_1(F_1(x_\delta) - F_1(x_{h*})) = p_1(F_1(x_\delta) - F_1(x')) < \varepsilon,\ c(x_\delta/a^h) - c(\tilde{x}/a^h) > \varepsilon$, and $\mathbb{E}[v_i^h(\mu(x_\delta), \mu_{-i})] \le \mathbb{E}[v_i^h(\mu(\tilde{x}), \mu_{-i})]$. Then

$$p_1 F_1^*(\tilde{x}) + \mathbb{E}[v_i^h(\mu(\tilde{x}), \mu_{-i})] - c\left(\frac{\tilde{x}}{a^h}\right) > p_1 F_1^*(x_\delta) + \mathbb{E}[v_i^h(\mu(x_\delta), \mu_{-i})] - c\left(\frac{x_\delta}{a^h}\right),$$

a contradiction. So we can conclude that $x_\ell^* \le x_{h*}$.

Also $x_\ell^* \le x_h^*$. If we assume otherwise, then we can find $x \in \{X_{i1}^\ell \setminus X_{i1}^h\}$ where $x > x_h^*$ and $\mu(x) = 0$. Lemma 2 rules out this possibility.

We have shown so far that $0 = x_{\ell*} \le x_{h*} \le x_\ell^* \le x_h^*$.

(3) For all $x \in (x_{\ell*}, x_{h*})$, $x \in BR_{i1}^\ell(\sigma_{-i})$ and for all $x \in (x_\ell^*, x_h^*)$, $x \in BR_{i1}^h(\sigma_{-i})$.

Given $x_{\ell*} < x_{h*}$, let $X_c^\ell = \{x | x \in (x_{\ell*}, x_{h*}) \setminus BR_{i1}^\ell(\sigma_{-i})\}$. If $x \in X_c^\ell$, then $\exists \varepsilon > 0$ such that for all $x' \in (x_{\ell,*}, x_{h,*}) \cap X_{i1}^\ell$,

$$p_1 F_1^*(x) + \mathbb{E}[v_i^\ell(\mu(x), \mu_{-i})] - c(x) < p_1 F_1^*(x') + \mathbb{E}[v_i^\ell(\mu(x'), \mu_{-i})] - c(x') - \varepsilon,$$

where $\mathbb{E}[v_i^\ell(\mu(x), \mu_{-i})] \ge \mathbb{E}[v_i^\ell(\mu(x'), \mu_{-i})]$ as $\mu(x') = 0$. Therefore $p_1 F_1^*(x) - c(x) < p_1 F_1^*(x') - c(x') - \varepsilon$, and for all $x' > x$, $p_1(F_1^*(x') - F_1^*(x)) > c(x') - c(x) - \varepsilon$.

Since $F_1^*(x)$ and $c(x)$ are continuous, then there is a $\delta_\varepsilon > 0$ such that for all $x' \in X_{i1}^\ell$, $|x' - x| \ge \delta_\varepsilon$. This implies that x is contained in an interval which is a subset of X_c^ℓ. Let a and b be the infimum and supremem of this interval respectively.

- If $b < x_{h*}$, then $\exists x' < x_{h*},\ x' \in X_{i1}^\ell$ where $|x' - b| < \delta, \forall \delta > 0$. Then, by the continuity of $F_1^*(x)$, $\exists x' \in X_{i1}^\ell$ and $p_1(F_1^*(x') - F_1^*(b)) < c(b) - c(\frac{a+b}{2})$. Then we know that

$$p_1 F_1^*(x') - p_1 F_1^*\left(\frac{a+b}{2}\right) < c(b) - c\left(\frac{a+b}{2}\right) \text{ and}$$
$$\mathbb{E}[v_i^\ell(\mu(x'), \mu_{-i})] \le \mathbb{E}\left[v_i^\ell\left(\mu\left(\frac{a+b}{2}\right), \mu_{-i}\right)\right],$$

  which contradicts $x' \in BR_{i1}^\ell(\sigma_{-i})$.

- If $b = x_{h*}$, then $\exists x' \in X_{i1}^h$, where $|x' - x_{h*}| < \delta$, $\forall \delta > 0$. We can take $x' \in X_{i1}^h$ such that $p_1(F_1^*(x') - F_1^*(x_{h*})) < c(\frac{b}{a^h}) - c(\frac{a+x_{h*}}{2a^h})$.
  - If $x' \notin X_{i1}^\ell$ then $\mu(x') = 1$, but since $\mathbb{E}[v_i^h(\mu(x'), \mu_{-i})] \le \mathbb{E}\left[v_i^h\left(\mu\left(\frac{a+x_{h*}}{2}\right), \mu_{-i}\right)\right]$, then this contradicts $x' \in BR_{i1}^h(\sigma_{-i})$.
  - If $x' \in X_1^\ell$, then $\mu(x') \in [0,1]$. If $\mu(x') \le \mu(\frac{a+x_{h*}}{2})$, then this contradicts $x' \in BR_{i1}^\ell(\sigma_{-i})$, but if $\mu(x') \ge \mu(\frac{a+x_{h*}}{2})$, this contradicts $x' \in BR_{i1}^h(\sigma_{-i})$.

Therefore X_c^ℓ must be empty.

Similarly, define $X_c^h = \{x | x \in (x_\ell^*, x_h^*) \setminus BR_{i1}^h(\sigma_{-i})\}$ and let $x \in X_c^h$. Then $\exists \delta_\varepsilon > 0$ such that for all $x' \in X_{i1}^h$, $|x' - x| \ge \delta_\varepsilon > 0$. Take a and b to be the infimum and supremum respectively of the interval of X_c^h containing x noting that $b < x_h^*$.

There is an $x' \in X^h_{i1}$ where $|x'-b| < \delta$ for all $\delta > 0$. Then we can take $x' \in BR^h_{i1}(\sigma_{-i})$ such that $p_1(F^*_1(x') - F^*_1(b)) < c(\frac{b}{a^h}) - c(\frac{b+a}{2a^h})$. This implies $p_1(F^*_1(x') - F^*_1(\frac{b+a}{2})) < c(\frac{x'}{a^h}) - c(\frac{b+a}{2a^h})$ and

$$\begin{aligned} p_1 F^*_1\left(\frac{b+a}{2}\right) - c\left(\frac{b+a}{2a^h}\right) + \mathbb{E}\left[v^h_i\left(\mu\left(\frac{b+a}{2}\right), \mu_{-i}\right)\right] \\ > p_1 F^*_1(x') - c\left(\frac{x'}{a^h}\right) + \mathbb{E}[v^h_i(\mu(x'), \mu_{-i})]. \end{aligned}$$

This contradicts $x' \in BR^h_{i1}(\sigma_{-i})$, and therefore X^h_c must be empty.

(4) Lastly, we show that $x_{h*} < x^*_\ell$, and for all $x \in (x_{h*}, x^*_\ell)$, $x \in BR^\ell_{i1}(\sigma_{-i}) \cap BR^h_{i1}(\sigma_{-i})$.

If $x^*_\ell = x_{h*}$, then $\forall \delta > 0$, there is $x_\ell \in X^\ell_{i1}$ and $x_h \in X^h_{i1}$ where $|x_h - x_\ell| < \delta$. By the continuity of $F^*_1(x)$ and $c(x)$, there is x_h and x_ℓ for which

$$p_1 F^*_1(x_h) - c\left(\frac{x_h}{a^h}\right) - \left(p_1 F^*_1(x_\ell) - c\left(\frac{x_\ell}{a^h}\right)\right) < \mathbb{E}[v^h_i(\mu(x_\ell), \mu_{-i})] - \mathbb{E}[v^h_i(\mu(x_h), \mu_{-i})]$$

since $\mu(x_\ell) = 0$, $\mu(x_h) = 1$, and $\mathbb{E}[v^h_i(0, \mu_{-i})] - \mathbb{E}[v^h_i(1, \mu_{-i})] > 0$. Then

$$p_1 F^*_1(x_\ell) - c\left(\frac{x_\ell}{a^h}\right) + \mathbb{E}[v^h_i(\mu(x_\ell), \mu_{-i})] > p_1 F^*_1(x_h) - c\left(\frac{x_h}{a^h}\right) + \mathbb{E}[v^h_i(\mu(x_h), \mu_{-i})],$$

which contradicts $x_h \in BR^h_{i1}(\sigma_{-i})$.

Define $X_c = \{x | x \in (x_{h*}, x^*_\ell) \setminus (BR^\ell_{i1}(\sigma_{-i}) \cup BR^h_{i1}(\sigma_{-i}))\}$. From Lemma 2, we know that for all $x' \in \{(x_{h*}, x^*_\ell) \cap (X^\ell_{i1} \cup X^h_{i1})\}$, $\mu(x') \in (0,1)$ as $\mu(x') = 1$, implies $x^*_\ell \leq x'$ and $\mu(x') = 0$ implies $x_{h*} \geq x'$. Therefore $x' \in X^\ell_{i1} \cap X^h_{i1}$.

Let $x \in X_c$ be given. Then for all $x', x'' \in \{(x_{h*}, x^*_\ell) \cap (X^\ell_{i1} \cap X^h_{i1})\}$ such that $x' < x < x''$ we must by Lemma C.3 have $\mu(x') \leq \mu(x'')$. Let $\mu^* \in [\sup\{\mu(x')\}, \inf\{\mu(x'')\}]$. These are well-defined as there is at least one such x' and x''.

If $\mu(x) \geq \mu^*$ then $\mathbb{E}[v^\ell_i(\mu(x), \mu_{-i})] \geq \mathbb{E}[v^\ell_i(\mu(x'), \mu_{-i})]$ for all x' and

$$\begin{aligned} p_1 F^*_1(x') - c(x') + \mathbb{E}[v^\ell_i(\mu(x'), \mu_{-i})] - \varepsilon_1 > p_1 F^*_1(x) - c(x) + \mathbb{E}[v^\ell_i(\mu(x), \mu_{-i})] \\ \Rightarrow p_1 F^*_1(x') - c(x') - \varepsilon_1 > p_1 F^*_1(x) - c(x). \end{aligned}$$

By continuity of $F^*_1(x)$ and $c(x)$, $\exists \delta_1 > 0$ such that $[x - \delta_1, x] \subset X_c$.

Similarly, if $\mu(x) < \mu^*$, then $\mathbb{E}[v^h_i(\mu(x), \mu_{-i})] \geq \mathbb{E}[v^h_i(\mu(x''), \mu_{-i})]$ for all x'' and

$$p_1 F^*_1(x'') - c\left(\frac{x''}{a^h}\right) - \varepsilon_2 > p_1 F^*_1(x) - c\left(\frac{x}{a^h}\right).$$

By continuity, $\exists \delta_2 > 0$ such that $[x, x + \delta_2] \subset X_c$. In either case, if $x \in X_c$, then there is an interval with some supremum b and infimum a such that $x \in (a, b) \subset X_c$.

If $b < x^*_\ell$, then there is an $\tilde{x} \in \{(x_{h*}, x^*_\ell) \cap X^\ell_{i1} \cap X^h_{i1}\}$ where $|\tilde{x} - b| < \delta$ for all $\delta > 0$, and therefore there is an $\tilde{x}$ where $p_1(F^*_1(\tilde{x}) - F^*_1(b)) < c(b/a^h) - c(\frac{b+a}{2a^h})$. It follows that $p_1(F^*_1(\tilde{x}) - F^*_1(\frac{b+a}{2})) < c(\tilde{x}/a^h) - c(\frac{b+a}{2a^h})$ and $p_1(F^*_1(\tilde{x}) - F^*_1(\frac{b+a}{2})) < c(\tilde{x}) - c(\frac{b+a}{2})$.

If $\mu((b+a)/2) < \mu(\tilde{x})$ then

$$p_1 F_1^*\left(\frac{b+a}{2}\right) - c\left(\frac{b+a}{2a^h}\right) + \mathbb{E}\left[v_i^h\left(\mu\left(\frac{b+a}{2}\right), \mu_{-i}\right)\right] > p_1 F_1^*(\tilde{x}) - c\left(\frac{\tilde{x}}{a^h}\right) + \mathbb{E}[v_i^h(\mu(\tilde{x}), \mu_{-i})].$$

If $\mu((b+a)/2) \geq \mu(\tilde{x})$ then

$$p_1 F_1^*\left(\frac{b+a}{2}\right) - c\left(\frac{b+a}{2}\right) + \mathbb{E}\left[v_i^\ell\left(\mu\left(\frac{b+a}{2}\right), \mu_{-i}\right)\right] > p_1 F_1^*(\tilde{x}) - c(\tilde{x}) + \mathbb{E}[v_i^\ell(\mu(\tilde{x}), \mu_{-i})].$$

In either case, this contradicts $\tilde{x} \in X_{i1}^\ell \cap X_{i1}^h$.

If $b = x_\ell^*$, then there is an $\tilde{x} \in X_{i1}^h$, such that $|\tilde{x} - b| < \delta$, and $\mu(\tilde{x}) = 1$. This implies that $p_1(F_1^*(\tilde{x}) - F_1^*(\frac{b+a}{2})) < c(\tilde{x}/a^h) - c(\frac{b+a}{2a^h})$, and

$$p_1 F_1^*\left(\frac{b+a}{2}\right) - c\left(\frac{b+a}{2a^h}\right) + \mathbb{E}\left[v_i^h\left(\mu\left(\frac{b+a}{2}\right), \mu_{-i}\right)\right] > p_1 F_1^*(\tilde{x}) - c\left(\frac{\tilde{x}}{a^h}\right) + \mathbb{E}[v_i^h(\mu(\tilde{x}), \mu_{-i})].$$

This contradicts $\tilde{x} \in X_{i1}^h$. Therefore X_c must be empty and for all $x \in (x_{h*}, x_\ell^*)$, we must have $x \in BR_{i1}^\ell(\sigma_{-i}) \cap BR_{i1}^h(\sigma_{-i})$. □

Lemma C.6. *In any SPBE, the belief function is continuous in output on $[0, x_h^*]$, is weakly increasing on (x_{h*}, x_ℓ^*), takes a value of zero for all $x \in [0, x_{h*}]$ when $x_{h*} > 0$, and takes a value of one for all $x \in [x_\ell^*, x_h^*]$ when $x_h^* > x_\ell^*$.*

Proof. To show that $\mu^*(x)$ is continuous on $(0, x_h^*)$, note that equilibrium expected payoffs of a low ability contestant are constant for all $x \in BR_{i1}^\ell(\sigma^*)$ and likewise for high ability contestants for all $x \in BR_{i1}^h(\sigma^*)$. Since $F_1^*(x)$ and $c(x)$ are continuous on $(0, \infty)$ and $\mathbb{E}[v_i^\ell(\mu^*(x), \mu_{-i})] = c(x) - p_1 F_1^*(x) + K^\ell(p_1, p_2)$ on $[0, x_\ell^*]$, then $\mathbb{E}[v_i^\ell(\mu^*(x), \mu_{-i})]$ must be continuous on this interval. Similarly, $\mathbb{E}[v_i^h(\mu^*(x), \mu_{-i})]$ is continuous on $[x_{h*}, x_h^*]$. Since $\mathbb{E}[v_i^h(\mu^*(x), \mu_{-i})]$ is strictly decreasing in $\mu^*(x)$, and $\mathbb{E}[v_i^\ell(\mu^*(x), \mu_{-i})]$ is strictly increasing in $\mu^*(x)$, then $\mu^*(x)$ must also be continuous on $BR_{i1}^\ell(\sigma^*) \cup BR_{i1}^h(\sigma^*) = [0, x_h^*]$.

To show the remaining properties of the equilibrium belief function, we first show that the set $[0, x_h^*] \setminus X_{i1}$ has no interior, i.e. there can be no interval $[a, b] \subset [0, x_h^*]$ where for all $x \in [a, b]$, $x \notin X_{i1}$. This implies that X_{i1} is dense in $[0, x_h^*]$.

If we let $[\tilde{a}, \tilde{b}] \subset [0, x_h^*] \setminus X_{i1}$ be given, then define a and b to be the infimum and supremum respectively of the interval in $[0, x_h^*] \setminus X_{i1}$ which contains $[\tilde{a}, \tilde{b}]$. Neither x_{h*} nor x_ℓ^* can be contained in the interval as they are the limit point of a subset of X_{i1}. Then the interval $[a, b]$ must be contained within either $[0, x_{h*}]$, $[x_{h*}, x_\ell^*]$, or $[x_\ell^*, x_h^*]$.

1. If $[a, b] \subset [0, x_{h*}]$, then for all $x \in [a, b]$, $x \in BR_{i1}^\ell(\sigma^*)$ and $F_1^*(x) = F_1^*(a)$. Therefore, $\mathbb{E}[v_i^\ell(\mu^*(b), \mu_{-i})] - c(b) = \mathbb{E}[v_i^\ell(\mu^*(a), \mu_{-i})] - c(a)$, which implies that

$\mu(b) > \mu(a)$. Since $\mu(x)$ is continuous, then for all $\delta > 0$, there is an $x \in X_{i1}^h$ such that $|x - b| < \delta$ and $\mu(x) > 0$. If $x \in X_{i1}^h \setminus X_{i1}^\ell$, then $\mu^*(x)$=1, and $x \notin BR_{i1}^h(\sigma^*)$, a contradiction. If $x \in X_{i1}^h \cap X_{i1}^\ell$ then depending on the value of $\mu^*((a+b)/2)$, it must be that either $x \notin BR_{i1}^h(\sigma^*)$ or $x \notin BR_{i1}^\ell(\sigma^*)$, again a contradiction.

2. If $[a, b] \subset [x_{h*}, x_\ell^*]$, then for all $x \in [a, b]$, $x \in \{BR_{i1}^\ell(\sigma^*) \cap BR_{i1}^h(\sigma^*)\}$ which implies

$$\mathbb{E}[v_i^\ell(\mu^*(b), \mu_{-i})] - c(b) = \mathbb{E}[v_i^\ell(\mu^*(a), \mu_{-i})] - c(a),$$
$$\mathbb{E}[v_i^h(\mu^*(b), \mu_{-i})] - c(b/a^h) = \mathbb{E}[v_i^h(\mu^*(a), \mu_{-i})] - c(a/a^h).$$

However, rearranging these equations, it is clear they cannot hold at the same time as the right hand sides are both strictly positive which contradicts Proposition 1.

$$\mathbb{E}[v_i^\ell(\mu^*(b), \mu_{-i})] - \mathbb{E}[v_i^\ell(\mu^*(a), \mu_{-i})] = c(b) - c(a)$$
$$\mathbb{E}[v_i^h(\mu^*(b), \mu_{-i})] - \mathbb{E}[v_i^h(\mu^*(a), \mu_{-i})] = c(b/a^h) - c(a/a^h)$$

3. If $[a, b] \subset [x_\ell^*, x_h^*]$, then for all $x \in [a, b]$, $x \in BR_{i1}^h(\sigma^*)$ and therefore,

$$\mathbb{E}[v_i^h(\mu^*(b), \mu_{-i})] - c(b/a^h) = \mathbb{E}[v_i^h(\mu^*(a), \mu_{-i})] - c(a/a^h),$$

and $\mu^*(b) < \mu^*(a) \leq 1$. Then for all $\delta > 0$, there is an $x \in X_{i1}^h$ such that $|x - b| < \delta$ and $\mu^*(x) = 1$. However, this contradicts the continuity of $\mu^*(x)$.

Now, if $x \in [0, x_{h*})$ and $\mu^*(x) = \varepsilon > 0$, then by the continuity of $\mu^*(x)$, $\exists \delta > 0$ where $\forall x', |x' - x| < \delta$, $\mu^*(x) > \varepsilon/2$. However for all $\delta > 0$ there is an $x' \in X_{i1}^\ell \setminus X_{i1}^h$ for which $\mu^*(x') = 0$, a contradiction. Therefore $\mu^*(x) = 0$ for all $x \in [0, x_{h*})$. Note that $\mu^*(x_{h*}) = 0$ when $x_{h*} > 0$, which follows from continuity from the left. Similarly, $\mu^*(x) = 1$ for all $x \in [x_\ell^*, x_h^*]$ when $x_\ell^* < x_h^*$. To show that $\mu^*(x)$ is weakly increasing on $[x_{h*}, x_\ell^*]$, let $x, y \in [x_{h*}, x_\ell^*]$ be such that, $\mu^*(x) > \mu^*(y)$ and $x < y$. Then there is an x' and y' arbitrarily close to x and y respectively, where $x', y' \in X_{i1}$ and therefore $\mu^*(x') \leq \mu^*(y')$. This is not consistent with $\mu^*(x)$ being continuous on $[0, x_h^*]$, a contradiction. □

Proof of Theorem 2

There are up to three distinct intervals in each equilibrium. We will show that the endpoints of these intervals and the distribution functions on the intervals are completely determined by the first order conditions of the contestants.

Conditions for x being in $BR_{i1}^h(\sigma^*)$ and $BR_{i1}^\ell(\sigma^*)$ are

$$BR_{i1}^h(\sigma^*): \; p_1 F_1^*(x) + \mathbb{E}[v_i^h(\mu^*(x), \mu_{-i})] - c\left(\frac{x}{a^h}\right) = K^h(p_1, p_2),$$
$$BR_{i1}^\ell(\sigma^*): \; p_1 F_1^*(x) + \mathbb{E}[v_i^\ell(\mu^*(x), \mu_{-i})] - c(x) = K^\ell(p_1, p_2) = 0.$$

For all values of $p_1 > 0$ and $p_2 > 0$, Lemma C.5 shows that $x_{h*} < x_\ell^*$, and therefore the interval $[x_{h*}, x_\ell^*]$ is non-trivial. On this interval, $x \in X_{i1}^\ell \cup X_{i1}^h$ implies $x \in X_{i1}^\ell \cap X_{i1}^h \subset BR_{i1}^\ell(\sigma^*) \cap BR_{i1}^h(\sigma^*)$. Subtracting the condition for $BR_{i1}^\ell(\sigma^*)$ from the condition for

$BR^h_{i1}(\sigma^*)$

$$\mathbb{E}[v_i^h(\mu^*(x), \mu_{-i})] - \mathbb{E}[v_i^\ell(\mu^*(x), \mu_{-i})] = c\left(\frac{x}{a^h}\right) - c(x) + K^h(p_1, p_2).$$

Taking the derivative of each side with respect to output gives (16):

$$\frac{d\mu^*(x)}{dx} d(\mu^*(x)) = c'(x) - \frac{1}{a^h} c'\left(\frac{x}{a^h}\right).$$

Note that on this interval, $\frac{d\mu^*(x)}{dx} > 0$ and therefore, $F^*_\mu(\mu^*(x)) = F^*_1(x)$.

Taking the derivative of the condition for X_1^ℓ and combining (16) we recover (17):

$$p_1 f_1^*(x) + \frac{d\mu^*(x)}{dx} d(\mu^*(x)) F_1^*(x) = c'(x)$$
$$\Rightarrow p_1 f_1^*(x) = c'(x)(1 - F_1^*(x)) + \frac{1}{a^h} c'\left(\frac{x}{a^h}\right) F_1^*(x).$$

From continuity of $F_1^*(x)$, $p_1 F_1^*(x_{h*}) = c(x_{h*})$. For a given x_{h*}, using the Picard - Lindelöf Theorem[24], we know that there is a unique solution for $f_1^*(x)$ on $[x_{h*}, x_\ell^*]$, and therefore $F_1^*(x)$ is determined on this interval.

To see why only one such x_{h*} can lead to an equilibrium, consider a different initial condition, $p_1 \tilde{F}^*(\tilde{x}_{h*}) = c(\tilde{x}_{h*})$ where $\tilde{x}_{h*} > x_{h*}$ and the associated $\tilde{f}_1^*(x)$ on $[\tilde{x}_{h*}, \tilde{x}_\ell^*]$. Then both $\tilde{F}_1^*(\tilde{x}_{h*}) > F_1^*(\tilde{x}_{h*})$ and $\tilde{\mu}^*(\tilde{x}_{h*}) < \mu^*(\tilde{x}_{h*})$, and for all $x \in [\tilde{x}_{h*}, x_\ell^*]$, $\tilde{F}_1^*(x) > F_1^*(x)$, $\tilde{f}_1^*(x) < f_1^*(x)$, and $\mu^*(x) > \tilde{\mu}^*(x)$. This implies that $\tilde{H}_1^*(x_\ell^*) = \int_0^{x_\ell^*} \tilde{\mu}^*(x) \tilde{f}_1^*(x) dx < \int_0^{x_\ell^*} \mu^*(x) f_1^*(x) dx = H_1^*(x_\ell^*)$ and therefore $\tilde{L}_1^*(x_\ell^*) > L_1^*(x_\ell^*) = 1$, a contradiction. Similarly, there cannot be an additional equilibrium where $\tilde{x}_{h*} < x_{h*}$.

The belief function on this interval is determined up to a constant by equation (16). The constant is determined by $\mu^*(x_{h,*})$ which is 0 when $x_{h*} > 0$, and needs to be characterized in equilibrium when $x_{h*} = 0$. Given this constant, the equilibrium strategies of high ability and low ability contestants can be constructed on this interval.

For small values of p_1 relative to p_2, this is the only non-trivial interval: $x_{h*} = 0$ and $x_\ell^* = x_h^*$. In this case, $\mu^*(x_{h*}) \in [0, \hat{\mu}]$ and $\mu^*(x_h^*) \in [\hat{\mu}, 1]$ both need to be determined in equilibrium along with x_h^*. By an argument similar to that for showing x_{h*} is unique, if $x_{h*} = 0$ then $\mu^*(x_{h*})$ is also uniquely determined. Then $\mu^*(x)$ and $F_1^*(x)$ are uniquely determined on this interval, and therefore x_h^* and $\mu^*(x_h^*)$ are also uniquely determined.

For larger p_1, $x_{h*} > 0$ and/or $x_h^* > x_\ell^*$. When the intervals are non-trivial, then the belief functions on these intervals were characterized in Lemma 5. Characterization of the output distributions directly follow. For $x \in [0, x_{h*})$, $\mathbb{E}[v_i^\ell(\mu^*(x), \mu_{-i})] = 0$ as $\mu^*(x) = 0$, and therefore $p_1 F_1^*(x) = c(x)$. For all $x \in [x_\ell^*, x_h^*]$, $\mathbb{E}[v_i^h(\mu^*(x), \mu_{-i})] = \mathbb{E}[v_i^h(1, \mu_{-i})]$ and $p_1 F_1^*(x) + \mathbb{E}[v_i^h(1, \mu_{-i})] = c(x/a^h) + K^h(p_1, p_2)$.

Given $F_1^*(x)$ and $\mu^*(x)$ on $[0, x^*]$, the output distribution of both the low and high ability contestants can be determined. Therefore $F_1^*(x), L_1^*(x)$ and $H_1^*(x)$ are uniquely characterized on X_{i1} where $\overline{X}_{i1} = [0, x_h^*]$. These distributions along with the second

[24] The right hand side of equation (17) is continuous in x and uniformly Lipshitz continuous in $F_1^*(x)$ on the interval of $[x_{h*}, x_\ell^*]$. Also, due to the properties of the cost function, the distribution function is bounded between 0 and 1.

period output distributions $L^*_{i2}(x|\eta_2)$ and $H^*_{i2}(x|\eta_2)$ form the unique SPBE.

Proposition C.3. *Let $F^*_\mu(M)$ be the equilibrium belief distribution associated with prize ratio p_1/p_2 and $\tilde{F}^*_\mu(M)$ be associated with $\tilde{p}_1/\tilde{p}_2$. Then $p_1/p_2 > \tilde{p}_1/\tilde{p}_2$ implies $F^*_\mu(M) <_{SOSD} \tilde{F}^*_\mu(M)$.*

Proof. Belief distributions that arise after the first contest for different prize structures must be equal at least at one point. If the distributions do not cross then one distribution FOSD the other and the distributions cannot have the same expected value. However, the expectation of the probability that a contestant is high ability is $\hat{\mu}$ in either case.

Let $\tilde{\mu}(\hat{\tilde{x}}) = \mu(\hat{x}) = \hat{M}$ be a point of intersection for belief distributions $\tilde{F}_\mu(M)$ and $F_\mu(M)$. Note that

$$f_\mu(\hat{M}) = \frac{\partial}{\partial \mu} F_1(\mu^{-1}(\hat{M})) = \frac{f_1(\mu^{-1}(\hat{M}))}{\mu'(\mu^{-1}(\hat{M}))} = \frac{f_1(\hat{x})}{\mu'(\hat{x})}.$$

From equations (16) and (17),

$$\frac{f_1(\hat{x})}{\mu'(\hat{x})} = \frac{d(\mu(\hat{x}))\left(c'(x) - F_1(\hat{x})\left(c'(x) - \frac{1}{a_h}c'\left(\frac{\hat{x}}{a_h}\right)\right)\right)}{p_1\left(c'(x) - \frac{1}{a_h}c'\left(\frac{\hat{x}}{a_h}\right)\right)} = \frac{d(\mu(\hat{x}))}{p_1}\left(\frac{a_h^\alpha}{a_h^\alpha - 1} - F_1(\hat{x})\right)$$
$$= \frac{p_2}{p_1}\left(\frac{a_h^\alpha(1 - F_1(\hat{x})) + F_1(\hat{x})}{a_h^\alpha}\right)$$

Because $\tilde{F}_1(\tilde{\mu}^{-1}(\hat{M})) = F_1(\mu^{-1}(\hat{M}))$, then $\tilde{f}_\mu(\hat{M}) \lesseqgtr f_\mu(\hat{M})$ when $\frac{\tilde{p}_2}{\tilde{p}_1} \lesseqgtr \frac{p_2}{p_1}$. Given that $\tilde{f}_\mu(\hat{M}) < f_\mu(\hat{M})$, as in the case when first contest prize is increased for a fixed second contest prize, this implies that $\tilde{F}_\mu(M)$ crosses $F_\mu(M)$ exactly once from above and $\tilde{F}_\mu(M) <_{SOSD} F_\mu(M)$. An increase the second contest prize for fixed first contest prize implies $\tilde{F}_\mu(M)$ crosses $F_\mu(M)$ exactly once from below and $\tilde{F}_\mu(M) >_{SOSD} F_\mu(M)$. □

C.2 Equilibrium construction

Assume the cost function takes the form, $c(x) = kx^\alpha$, with $\alpha \geq 1$ and $k > 0$ and let $\hat{\mu} = 1/2$. For the equilibrium of the first contest we find the ex-ante expected distribution of each contestant over each of the potential three ranges of output which depend on the values of p_1 and p_2. Let $A = \frac{a^{h^\alpha}}{a^{h^\alpha}-1}$.

For any values of p_1 and p_2, $x_{h*} < x^*_\ell$. For $x \in [x_{h*}, x^*_\ell]$ the expected output distribution satisfies equation (17). The family of solutions is

$$F^*_1(x) = Be^{(c(x/a^h) - c(x))/p_1} + A,$$

with boundary condition $F_1^*(x_{h*}) = \frac{1}{p_1} k x_{h*}^\alpha$. The solution is

$$F_1^*(x) = A - \left(A - \frac{1}{p_1} k x_{h*}^\alpha\right) e^{-\frac{1}{Ap_1}(kx^\alpha - kx_{h*}^\alpha)}.$$

The belief function satisfies the condition in equation (16) which simplifies under this parameterization to $p_2 \mu'(x) = c'(x)$. The belief function is $\mu^*(x) = \frac{1}{p_2}(kx^\alpha + C)$, where $C = -c(x_{h*})$ if $x_{h*} > 0$ and $C = p_2 \mu^*(x_{h*})$ if $x_{h*} = 0$. Therefore

$$F_1^*(x) = A - \left(A - \frac{1}{p_1} k x_{h*}^\alpha\right) e^{-\frac{p_2}{Ap_1}(\mu^*(x) - \mu^*(x_{h*}))},$$

where $\mu^*(x_{h*}) = 0$ when $x_{h*} > 0$.

If $x_{h*} > 0$, then $F_1^*(x) = \frac{1}{p_1} k x^\alpha$ and $\mu^*(x) = 0$ for $x \in [0, x_{h*}]$. If $x_\ell^* < x_h^*$, then $F_1^*(x) = \frac{1}{p_1}\left(\frac{1}{a^{h^\alpha}} k x^\alpha + K^h(p_1, p_2) - \mathbb{E}[v_i^h(1, \mu_{-i})]\right)$ and $\mu^*(x) = 1$ for $x \in [x_\ell^*, x_h^*]$.

Given $F_1^*(x)$ and $\mu^*(x)$, the output distribution of the both the high and low ability contestants comes from using $2F_1^*(x) = L_1^*(x) + H_1^*(x)$ and $\mu^*(x) = \frac{h_1^*(x)}{\ell_1^*(x) + h_1^*(x)}$.

Over the range $x \in [x_{h,*}, x_\ell^*]$ these distributions are

$$H_1^*(x) = 2(A - \frac{1}{p_1} k x_{h,*}^\alpha)\left(\frac{Ap_1}{p_2} - (\mu^*(x) + \frac{Ap_1}{p_2}) e^{-\frac{p_2}{Ap_1}(\mu^*(x) - \mu^*(x_{h,*}))}\right) + 2A\mu^*(x_{h,*}) \quad \text{and}$$

$$L_1^*(x) = 2A(1 - \mu^*(x_{h,*})) + 2(A - \frac{1}{p_1} k x_{h,*}^\alpha)\left((\mu^*(x) + \frac{Ap_1}{p_2} - 1) e^{-\frac{p_2}{Ap_1}(\mu^*(x) - \mu^*(x_{h,*}))} - \frac{Ap_1}{p_2}\right).$$

Small prize in first contest

Given p_2, for p_1 close enough to 0 (specifically for $p_1 < \frac{p_2}{2}((A^2 - A)\log(a^{h^\alpha}) - A)^{-1}$), both $x_{h*} = 0$ and $x_\ell^* = x_h^*$. The expected output distribution becomes

$$F_1^*(x) = A\left(1 - e^{-\frac{p_2}{Ap_1}(\mu^*(x) - \mu^*(x_{h*}))}\right), \text{ for } 0 \leq x \leq x_h^*.$$

Given $H_1^*(x_{h*}) = F_1^*(x_{h*}) = F_1^*(0) = 0$, the output distribution of the high ability contestant is

$$H_1^*(x) = \int_0^x \mu^*(t) f_1^*(t) dt = 2F_1^*(x)\left(\mu^*(x) + \frac{Ap_1}{p_2}\right) - 2A(\mu^*(x) - \mu^*(x_{h*})).$$

Combining $F_1^*(x_h^*) = 1$ and $H_1(x_h^*) = 1$ gives

$$\mu^*(x_h^*) - \mu^*(x_{h*}) = \frac{p_1}{p_2} + \frac{2\mu^*(x_h^*) - 1}{2A}.$$

Plugging back into $F_1^*(x_h^*) = 1$, we can solve the belief function at each end point:

$$\mu^*(x_h^*) = \frac{1}{2} + \frac{p_1}{p_2}(A^2 \log(a^{h^\alpha}) - A) \text{ and } \mu^*(x_{h*}) = \frac{1}{2} + \frac{p_1}{p_2}((A^2 - A)\log(a^{h^\alpha}) - A).$$

Therefore $\mu^*(x_h^*) - \mu^*(x_{h*}) = \frac{p_1}{p_2} A \log(a^{h^\alpha})$ and $k{x_h^*}^\alpha = p_1 A \log(a^{h^\alpha})$.

Example of parameters that fall in this category: $c(e) = e^2$, $a^h = 2$ and $p_1 = .5$ and $p_2 = 1$.

Intermediate prize in first contest

For larger p_1 compared to p_2, (specifically $p_1 > \frac{p_2}{2}((A^2 - A)\log(a^{h^\alpha}) - A)^{-1})$, then $H_1^*(x_\ell^*) < 1$ and $x_\ell^* < x_h^*$. The expected output distribution is

$$F_1^*(x) = \begin{cases} A\left(1 - e^{\frac{-p_2}{Ap_1}(\mu^*(x) - \mu^*(x_{h*}))}\right) & 0 \le x \le x_\ell^* \\ \frac{1}{p_1}(\frac{k}{a^{h^\alpha}} x^\alpha + K^h(p_1,p_2) - \mathbb{E}[v_i^h(1,\mu_{-i})]) & x_\ell^* \le x \le x_h^* \end{cases}.$$

The output distributions of the high and low ability contestants are

$$H_1^*(x) = 2F_1^*(x)\left(\mu^*(x) + \frac{Ap_1}{p_2}\right) - 2A(\mu^*(x) - \mu^*(x_{h*})) \text{ and}$$

$$L_1^*(x) = 2F_1^*(x)\left(1 - \mu^*(x) - \frac{Ap_1}{p_2}\right) + 2A(\mu^*(x) - \mu^*(x_{h*})).$$

To characterize the equilibrium we need to solve for $\mu^*(0)$, x_ℓ^*, x_h^*, and $K^h(p_1,p_2) - \mathbb{E}[v_i^h(1,\mu_{-i})]$.

1. Continuity of the belief function: $\mu^*(x_\ell^*) = 1$ implies that $k{x_\ell^*}^\alpha = p_2(1 - \mu^*(x_{h*}))$:

2. $L_1^*(x_\ell^*) = 1$ gives an implicit equation for $\mu^*(x_{h*})$.

$$1 = 2A\left((1 - \mu^*(x_{h*})) - \frac{Ap_1}{p_2}\left(1 - e^{-\frac{p_2}{Ap_1}(1-\mu^*(x_{h*}))}\right)\right)$$

3. By continuity of $F_1^*(x)$ at x_ℓ^*, we can find $K^h(p_1,p_2) - \mathbb{E}[v_i^h(1,\mu_{-i})]$:

$$A - Ae^{-\frac{p_2}{Ap_1}(1-\mu^*(x_{h*}))} = \frac{1}{p_1}(\frac{k}{a^{h^\alpha}}{x_\ell^*}^\alpha + K^h(p_1,p_2) - \mathbb{E}[v_i^h(1,\mu_{-i})]).$$

   Using the equation that determines $\mu^*(x_{h*})$ and the belief equations, this simplifies to

$$K^h(p_1,p_2) - \mathbb{E}[v_i^h(1,\mu_{-i})] = \frac{p_2}{A}\left(\frac{1}{2} - \mu(x_{h*})\right).$$

4. From $F_1^*(x_h^*) = 1$ we can find x_h^*:

$$k{x_h^*}^\alpha = p_1 a^{h^\alpha} - p_2(a^{h^\alpha} - 1)(1/2 - \mu^*(x_{h*})).$$

Example of parameters that fall in this category: $c(e) = e^2$, $a^h = 2$ and $p_1 = 0.8$ and $p_2 = 1$.

Large prize in first contest

For large enough p_1, all three intervals are non-trivial, $\mu^*(x_{h*}) = 0$ and $\mu^*(x_\ell^*) = 1$. The distribution functions are

$$F_1^*(x) = \begin{cases} \frac{1}{p_1} k x^\alpha & 0 \le x \le x_{h*} \\ A - (A - \frac{1}{p_1} k x_{h*}^\alpha) e^{\frac{-p_2}{Ap_1}\mu^*(x)} & x_{h*} \le x \le x_\ell^* \\ \frac{1}{p_1}(\frac{k}{a^{h\alpha}} x^\alpha + K^h(p_1,p_2) - \mathbb{E}[v_i^h(1,\mu_{-i})]) & x_\ell^* \le x \le x_h^* \end{cases},$$

$$L_1^*(x) = \begin{cases} \frac{2}{p_1} k x^\alpha, & 0 \le x \le x_{h*} \\ 2A + 2(A - \frac{1}{p_1} k x_{h*}^\alpha)\left((\mu^*(x) + \frac{Ap_1}{p_2} - 1) e^{-\frac{p_2}{Ap_1}\mu^*(x)} - \frac{Ap_1}{p_2}\right), & x_{h*} \le x \le x_\ell^* \\ 1, & x_\ell^* \le x \le x_h^* \end{cases}, \text{ and}$$

$$H_1^*(x) = \begin{cases} 0, & 0 \le x \le x_{h*} \\ 2(A - \frac{1}{p_1} k x_{h*}^\alpha)\left(\frac{Ap_1}{p_2} - (\mu^*(x) + \frac{Ap_1}{p_2}) e^{-\frac{p_2}{Ap_1}\mu^*(x)}\right), & x_{h*} \le x \le x_\ell^* \\ \frac{2}{p_1}(\frac{k}{a_h^\alpha} x^\alpha + K^h(p_1,p_2) - \mathbb{E}[v_i^h(1,\mu_{-i})]) - 1 & x_\ell^* \le x \le x_h^* \end{cases}.$$

Using $\mu^*(x_\ell^*) = 1$ and $L_1^*(x_\ell^*) = 1$ identifies the endpoints of the middle interval:

$$k x_{h*}^\alpha = p_1 \left(A - \frac{(2A-1)p_2}{2Ap_1(1 - e^{-\frac{p_2}{Ap_1}})}\right) \text{ and } k x_\ell^{*\alpha} = p_2 + p_1 \left(A - \frac{(2A-1)p_2}{2Ap_1(1 - e^{-\frac{p_2}{Ap_1}})}\right).$$

Continuity of the expected output distribution at x_ℓ^* gives

$$\frac{1}{p_1}(\frac{1}{a^{h\alpha}} k (x_\ell^*)^\alpha + K^h(p_1,p_2) - \mathbb{E}[v_i^h(1,\mu_{-i})]) = A - (A - \frac{1}{p_1} k x_{h*}^\alpha) e^{-\frac{p_2}{Ap_1}}.$$

Then the constant associated with the third interval is

$$K^h(p_1,p_2) - \mathbb{E}[v_i^h(1,\mu_{-i})] = p_1 - \frac{p_2}{a_h^\alpha}\left(1 + \frac{2A-1}{2A} \frac{a^{h\alpha} e^{-\frac{p_2}{Ap_1}} - 1}{1 - e^{-\frac{p_2}{Ap_1}}}\right).$$

Using $F_1^*(x_h^*) = 1$ the endpoint of the upper interval is characterized by

$$k x_h^{*\alpha} = a^{h\alpha}\left(p_1 - (K^h(p_1,p_2) - \mathbb{E}[v_i^h(1,\mu_{-i})])\right) = p_2 \left(1 + \left(\frac{2A-1}{2A}\right) \frac{a^{h\alpha} e^{-\frac{p_2}{Ap_1}} - 1}{1 - e^{-\frac{p_2}{Ap_1}}}\right).$$

Example of parameters that fall in this category: $c(e) = e^2$, $a^h = 2$ and $p_1 = 1$ and $p_2 = 1$.

C.2.1 Second Stage Contest

As derived in the proof of Proposition 7, for given beliefs μ_w and μ_s, the expected output distribution of the weak and strong contestants are

$$L^*_{s2}(x) = \begin{cases} \frac{kx^\alpha}{p_2(1-\mu_s)}, & 0 \le x \le x^*_s \\ 1, & x^*_s \le x \le x^* \end{cases}, \quad H^*_{s2}(x) = \begin{cases} 0, & 0 \le x \le x^*_s \\ \frac{kx^\alpha - kx^{*\alpha}_s}{p_2\mu_s}, & x^*_s \le x \le x^*_w \\ 1 - \frac{kx^{*\alpha} - kx^\alpha}{a^{h^\alpha} p_2 \mu_w}, & x^*_w \le x \le x^* \end{cases},$$

$$L^*_{w2}(x) = \begin{cases} \frac{kx^\alpha}{p_2(1-\mu_w)} + \frac{\mu_s - \mu_w}{1-\mu_w}\left(\frac{a^{h^\alpha}-1}{a^{h^\alpha}}\right), & 0 \le x \le x^*_s \\ 1 - \frac{kx^{*\ \alpha}_w - kx^\alpha}{a^{h^\alpha} p_2(1-\mu_w)}, & x^*_s \le x \le x^*_w \\ 1, & x^*_w \le x \le x^* \end{cases}, \text{ and}$$

$$H^*_{w2}(x) = \begin{cases} 0, & 0 \le x \le x^*_w \\ 1 - \frac{kx^{*\alpha} - kx^\alpha}{a^{h^\alpha} p_2 \mu_w}, & x^*_w \le x \le x^* \end{cases}.$$

The expected output distributions are characterized by

$$F^*_{s2}(x) = \begin{cases} \frac{k}{p_2} x^\alpha, & 0 \le x \le x^*_w \\ 1 - \frac{kx^{*\alpha} - kx^\alpha}{a^{h^\alpha} p_2}, & x^*_w \le x \le x^* \end{cases} \quad \text{and}$$

$$F^*_{w2}(x) = \begin{cases} \frac{k}{p_2} x^\alpha + \left(\frac{a^{h^\alpha}-1}{a^{h^\alpha}}\right)(\mu_s - \mu_w), & 0 \le x \le x^*_s \\ 1 - \frac{kx^{*\alpha} - kx^\alpha}{a^{h^\alpha} p_2}, & x^*_s \le x \le x^* \end{cases},$$

where

$$\begin{aligned} kx^{*\ \alpha}_w &= p_2(1-\mu_w), \\ kx^{*\alpha}_s &= p_2(1-\mu_s), \text{ and} \\ kx^{*\alpha} &= p_2(1-\mu_w) + p_2\mu_w a^{h^\alpha}. \end{aligned}$$