# Repeated Contests with Private Information

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# Contests

Contests are used to motivate effort by giving a prize to the best contestant

- Firms compete to earn patents or awards
- Employees compete for bonus or promotion
- All-pay auctions

Contestants may have information about their own ability to win the contest

- R&D costs
- Worker productivity
- Value

Contestants use information to choose optimal effort

# Dynamic Contests

Contests are often dynamic

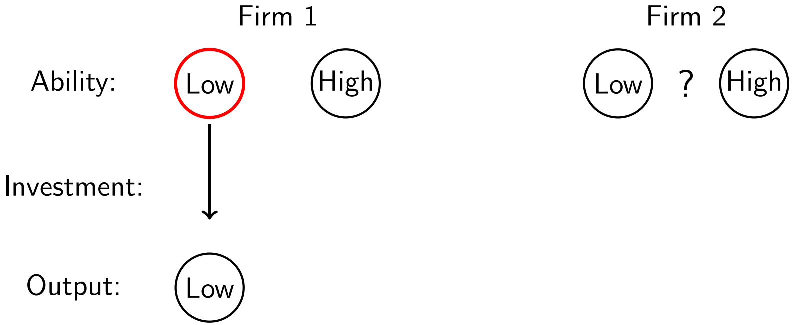
- Repeated interactions
- Occur over a period of time

Effort choices can reveal information about ability

- Information can be used later in contest
- How does this affect effort choices and outcomes of contest?
- Do contestants want to reveal ability?

# Firms Competing for Research Prize

Single contest, opponents ability unknown.

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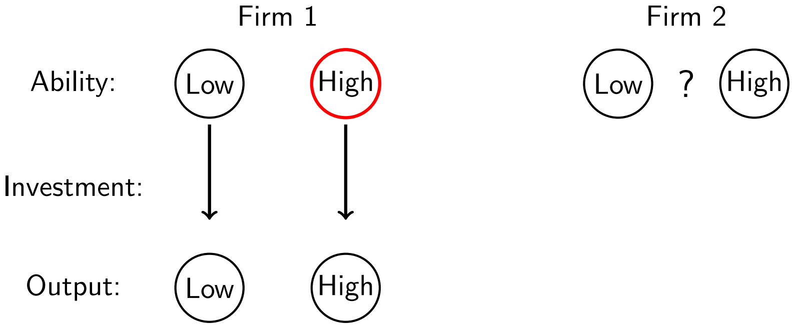


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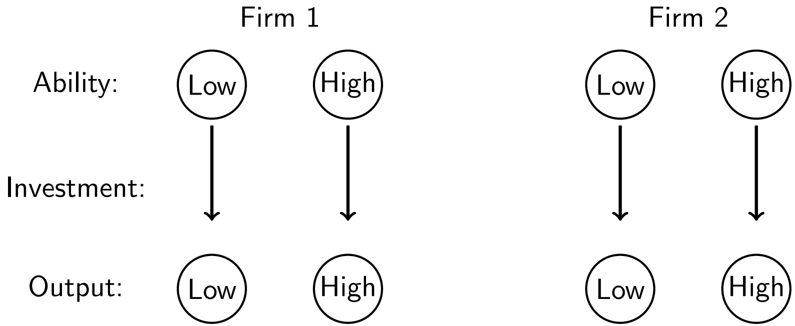
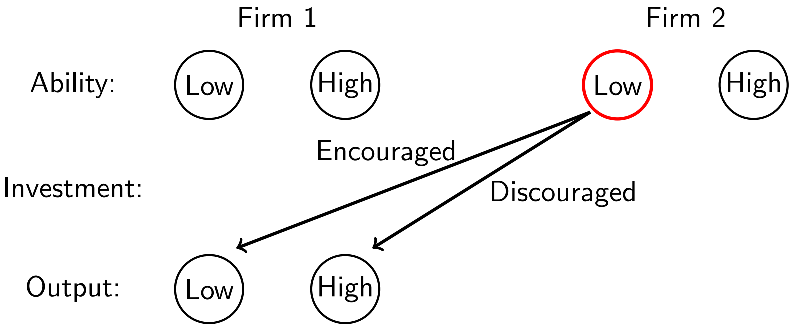



- If Firm 1 has low ability, chooses low investment and produces low output.
- If Firm 1 has high ability, produces high output.
- A firm with high ability will win the contest against a low ability firm.

# Firms Competing for Research Prize

How would Firm 2 respond to information from first contest:

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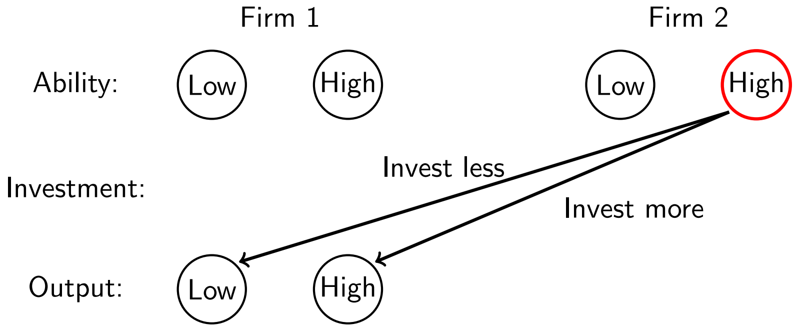
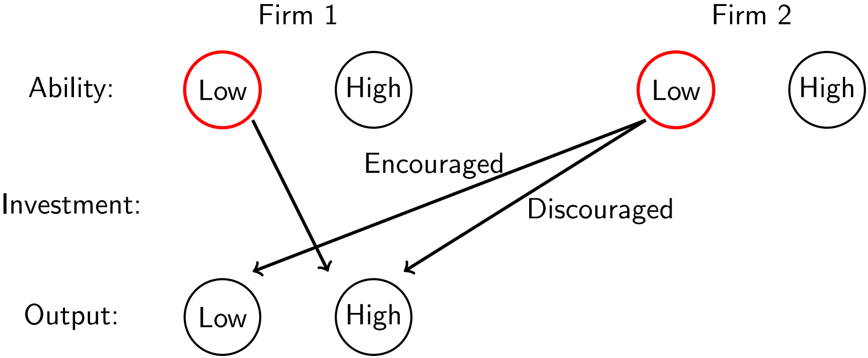



- If Firm 2 has low ability, then they are encouraged if they know Firm 1 also has low ability, but discouraged if Firm 1 has high ability.
- If Firm 2 has high ability, then they invest less if Firm 1 has low ability, and invest more if Firm 1 also has high ability.

# Firms Competing for Research Prize

Strategic effects of a second contest




- *Bluffing* - A firm with low ability wants to appear strong.

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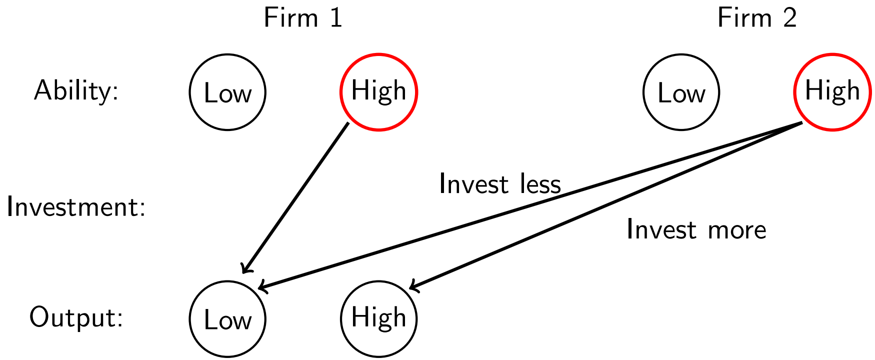


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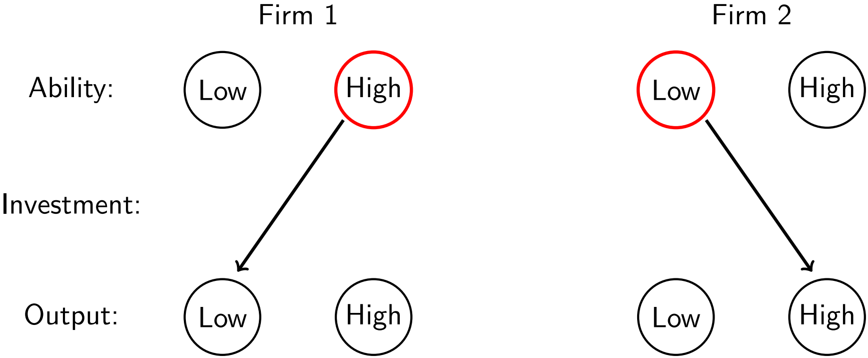


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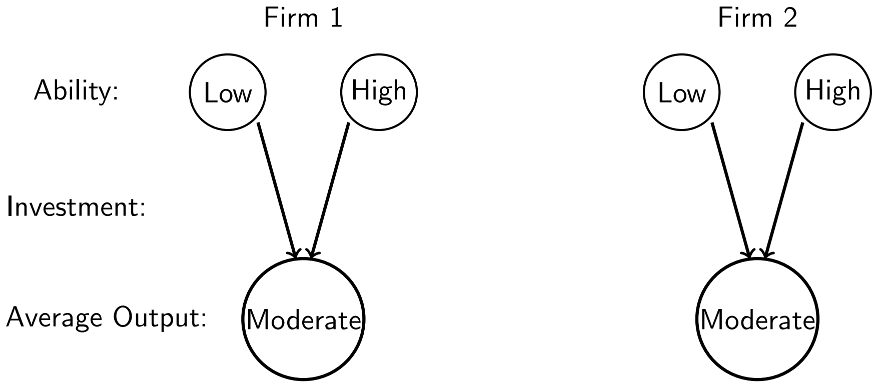


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Unique symmetric equilibrium

# Preview of Results

Comparison of repeated contest to single contest benchmark:

1. A high ability contestant may lose to a low ability contestant.
2. Expected aggregate output is reduced.
   - Strategic effects: in first contest, low ability players effort is higher and high ability players effort is lower.
   - After output is revealed, competitiveness is often reduced.
3. Expected payoffs of high ability contestants are higher, while low ability contestants the same.

# Selected Literature

Dynamic contests with asymmetric information:

- All-pay contests - Hörner and Sahuguet (2007), Münster (2009)

Contest design

- Competitions - Che and Gale (2003), Ederer (2010), Ridlon and Shin (2013)
- Tournaments - Rosen (1986), Fullerton and McAfee (1999), Moldovanu and Sela (2006), Ye (2007)
- First price auctions - Ortega Reichert (2001), Bergemann and Hörner (2010)

Single contest benchmark - Siegel (2014)

# Stage Game

- Two contestants, $i = 1, 2$
- Independently endowed with ability, $a^i \in \{a_\ell, a_h\}$
- $Pr(a^1 = a_h) = \mu_1$ and $\Pr(a^2 = a_h) = \mu_2$.
- Ability is private information for each player.
- Cost of effort: $c(e^i)$, $c'(e^i) > 0$, $c''(e^i) \geq 0$.
- Output: $x(a^i, e^i) = a^i \cdot e^i$
- Payoffs in terms of output

$$E[\pi^i(x^i, a^i)] = \Pr(x^{-i} < x^i) + \frac{1}{2}\Pr(x^{-i} = x^i) - c(x^i/a^i)$$

# Single Contest Benchmark

Timeline

1. Probabilities commonly known: $i = s, w$ where $\mu_s \geq \mu_w$
2. Abilities privately realized: $(a^s, a^w)$
3. Efforts chosen: $(e^s, e^w)$
4. Outputs realized: $(x^s, x^w)$, winner determined

# Equilibrium Best Response Sets

**Proposition 1**

There is a unique Bayesian Nash equilibrium in mixed strategies where, for each player,

1. the output distributions are continuous (except maybe at zero)
2. best response sets are disjoint, monotonic, and have no gaps.

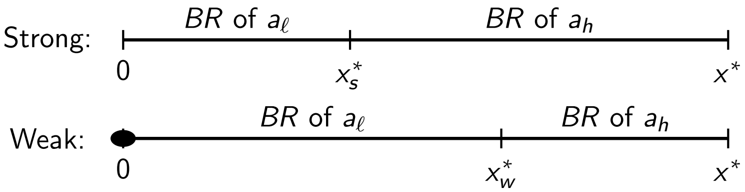


Figure 1: Best response sets of the strong and weak players.

# Equilibrium Output Distributions

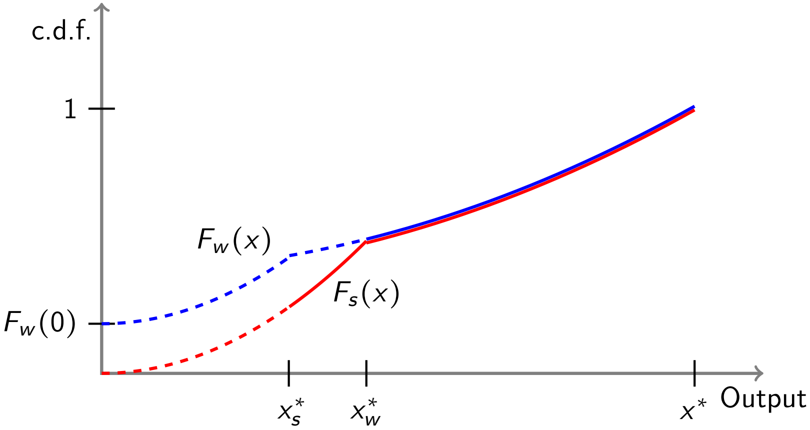


Figure 2: Expected output distributions of contestants in a single contest with quadratic costs.

# Ex-interim Equilibrium Payoffs

| | High Ability | Low Ability |
|---|---|---|
| Strong | $1 - c(x^*/a_h)$ | $F_w(0)$ |
| Weak | $1 - c(x^*/a_h)$ | 0 |

# Ex-interim Equilibrium Payoffs

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- Low ability contestant expects positive payoff only when strong contestant
- High ability contestant expects same payoff as strong or weak contestant
  - Decreases with the competitiveness of the contest

# Repeated Contests

Timeline:

1. Probabilities commonly known, $\mu_1 = \mu_2 = 1/2$
2. Abilities privately realized: $(a^1, a^2)$
3. Stage game is played.
4. Beliefs about opponent's ability updated: $(\mu(x_1^1), \mu(x_1^2))$
5. Stage game is played.

Contestants maximize the sum of their payoffs over the two contests.

- Do not discount future payoffs.
- No direct discouragement effect.

# Repeated Contests

## Definition (Symmetric Perfect Bayesian Equilibrium)

A set of output distributions and belief function $\{H_1^i(x_1), L_1^i(x_1), H_2^i(x_2|\mu_i, \mu_{-i}), L_2^i(x_2|\mu_i, \mu_{-i}), \mu(x)\}$ form a SPBE if

1. interim strategies are the same for each contestant,
2. contestants update beliefs about opponent's ability using Bayes' rule,
3. for every set of beliefs after first contest, strategies in the second contest are the unique Bayesian equilibrium of the single contest benchmark, and
4. given 1,2, and 3, contestants choose first period output to maximize payoffs over two contests. Formal

# Countervailing Incentives

## Proposition 2

The expected value of second contest increases in opponent's belief for low ability contestants and decreases in opponent's belief for high ability contestants. ▸ Proof

Second contest payoffs

| | High Ability | Low Ability |
|---|---|---|
| Strong | $1 - c(x^*/a_h)$ | $F_w(0)$ |
| Weak | $1 - c(x^*/a_h)$ | $0$ |

- Value of $x^*$ increasing in $\min\{\mu(x_1^1), \mu(x_1^2)\}$.
- Value of $F_w(0)$ increasing in $\mu(x_1^1) - \mu(x_1^2)$.

# Repeated Contest Equilibrium

### Theorem (Uniqueness of Equilibrium)

*There is a unique SPBE where in the first contest*

1. *the output distributions are continuous,*
2. *the belief function is continuous and weakly increasing in output, and*
3. *best response sets for high and low ability contestants are intervals with non-trivial overlap.*

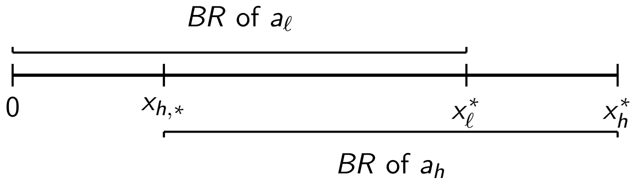
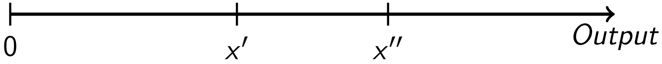



Figure 3: Best response sets in the first contest.

# Monotonic Beliefs

**Lemma 1**

In every SPBE, $\mu(x)$ is weakly increasing in output over the support of first period strategies.




$$0 \leq \mu(x'') < \mu(x') \leq 1$$

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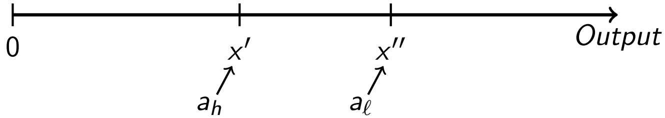


$$0 \leq \mu(x'') < \mu(x') \leq 1$$

- Current marginal costs (as in single contest)
- Value of the second contest (countervailing incentives)

# Overlapping Best Response Sets

**Lemma 2**

The best response sets of high ability and low ability contestants are intervals whose intersection is non-empty.

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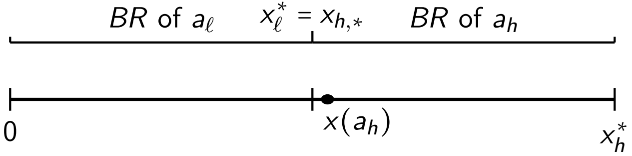


$$\mu(x(a_h)) = 1$$

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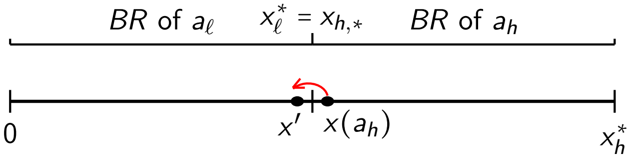


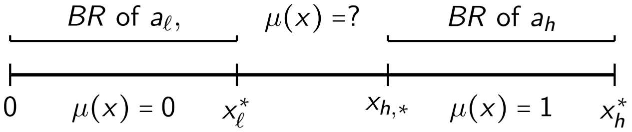

$$\mu(x(a_h)) = 1$$

$$\mu(x') = 0$$

Payoffs continuous in first contest, discrete jump in expected payoffs in second contest

# Best Response Sets are Intervals

No gap between best response sets



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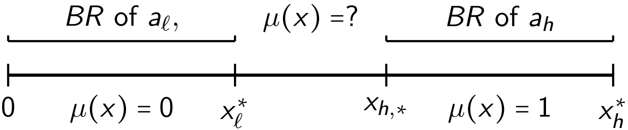


- If $\mu(x) \approx 1$, then low ability players prefer outputs off equilibrium path
- If $\mu(x) \approx 0$, then high ability players prefer outputs off equilibrium path

# Best Response Sets are Intervals

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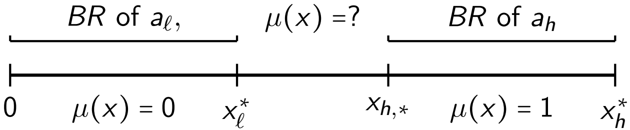



- If $\mu(x) \approx 1$, then low ability players prefer outputs off equilibrium path
- If $\mu(x) \approx 0$, then high ability players prefer outputs off equilibrium path

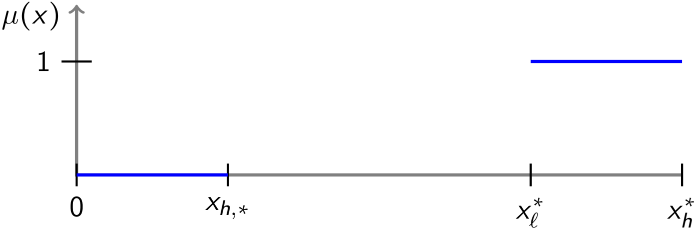
No beliefs off equilibrium path that punish both high and low ability contestants.

- No gap within best response sets
- Unique equilibrium

# Equilibrium Strategies and Beliefs


- For $0 \leq x < x_{h,*}$, low ability players indifferent: $F_1^*(x) = c(x)$.

# Equilibrium Strategies and Beliefs




- For $0 \leq x < x_{h,*}$, low ability players indifferent: $F_1^*(x) = c(x)$.
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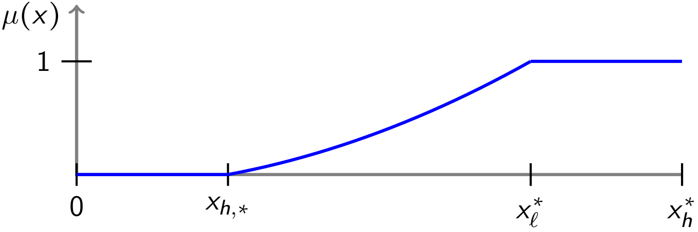
# Equilibrium Strategies and Beliefs




- For $0 \leq x < x_{h,*}$, low ability players indifferent: $F_1^*(x) = c(x)$.
- For $x_\ell^* < x \leq x_h^*$, high ability players indifferent: $F_1^*(x) = c(x_1^i/a_h) + K$.
- For $x_{h,*} \leq x \leq x_\ell^*$, both low and high ability contestants indifferent. Belief function must increases to offset differences in marginal cost.

# Bluffing and Sandbagging

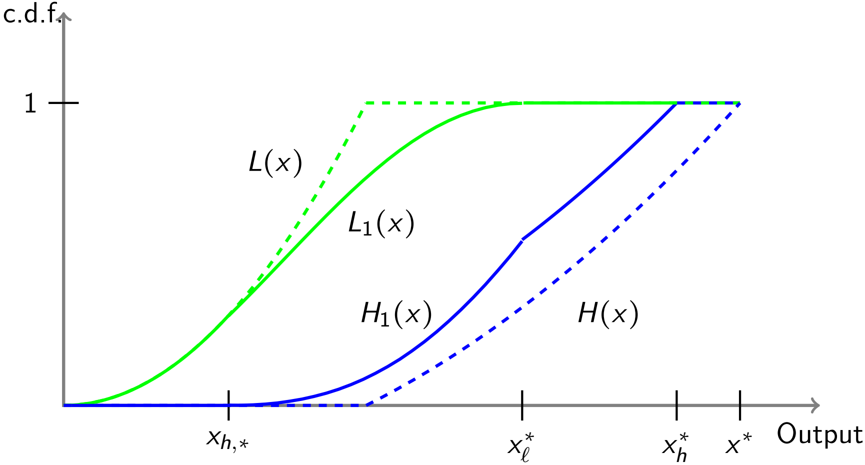



Figure 4: Distribution of strategies in the first of two contests compared to benchmark strategies. $(a_h = 2 \text{ and } c(e) = \frac{1}{2}e^2)$

# Surprise Victories

**Corollary 1**

A low ability player has a positive probability of winning each contest, even if they are competing against a high ability player.

- In symmetric benchmark, high ability contestant always wins against a low ability contestant
  - Used to justify efficient entry into later rounds of tournaments
- High ability players want to avoid ramping up competition
  - May lose earlier round to lesser opponent
  - Competition may be dampened in later round

# Reduced Expected Output

**Corollary 2**

Given the cost effort, $c(e) = ke^{\alpha}$, $k > 0, \alpha \geq 1$, average expected output of each contest is less than the expected output of a single contest. ▸ Proof

- In the first contest, lowered output of high ability player offsets the increased output of the low ability player
- In the second contest, increased information often reduces competition
  - A mid-term evaluation during a dynamic competition may reduce overall productivity
- Positive note: average expected payoff of high ability player higher than in single contest.
  - Lobbying, war of attrition

# Results - Reduced Output

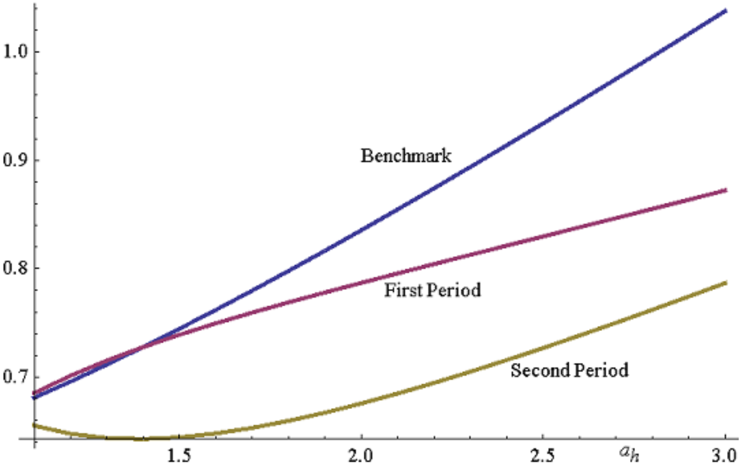



Figure : Output in terms of ability ratio, cost: $c(e) = e^2$.

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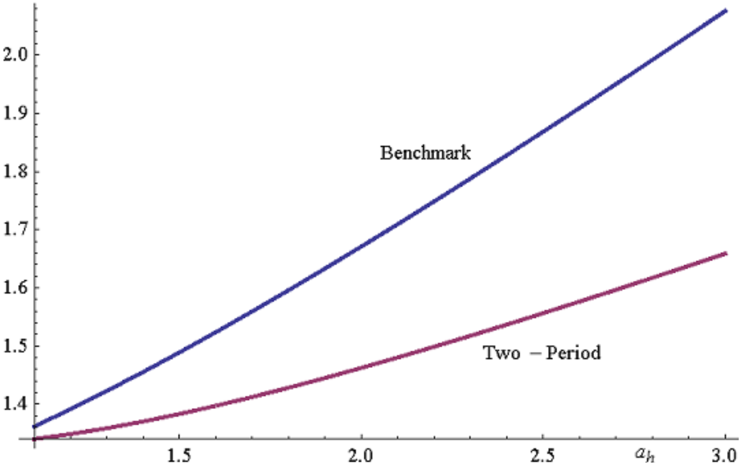


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# Results - Increased Payoffs

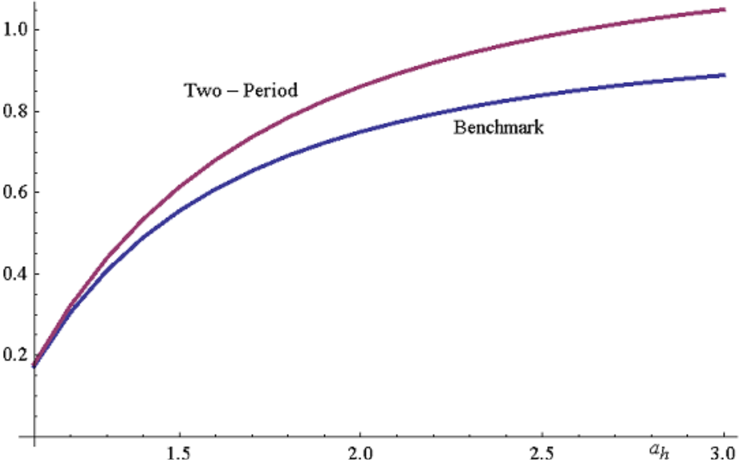


Figure : Payoffs of high ability player in terms of ability ratio, cost: $c(e) = e^2$.

# Conclusion

In unique equilibrium:

- Both high and low ability players benefit from hiding information about their ability.
  - Sandbagging and Bluffing
- Low ability players may beat high ability players.
- Lower average output and a higher reward for players with higher ability.
  - Increase information before game starts, or prevent information release during competition
  - Decrease value of second contest relative to the first contest.

Generalizations of payoff structure

- Contests have different prizes
- Contestants must win first contest to be eligible for second contest.

# The End

Thanks!

# Repeated Contests

## Definition (Symmetric Perfect Bayesian Equilibrium)

A set of output distributions $\{H_1^i(x_1), L_1^i(x_1), H_2^i(x_2|\mu_i, \mu_{-i}), L_2^i(x_2|\mu_i, \mu_{-i}) \text{ for } i = 1, 2\}$ form a SPBE if

1. $H_1^1(x) = H_1^2(x)$, $L_1^1(x) = L_1^2(x)$,
2. $\mu_i = \mu(x_1^i) = \frac{h_1(x_1^i)}{h_1(x_1^i)+\ell_1(x_1^i)}$, for $i = 1, 2$,
3. for every $(\mu_i, \mu_{-i})$,

$$(H_2^i(x|\mu_i, \mu_{-i}), L_2^i(x|\mu_i, \mu_{-i})) = \begin{cases} (H_w^*(x|\mu_i, \mu_{-i}), L_w^*(x|\mu_i, \mu_{-i})), & \text{if } \mu_i \leq \mu_{-i} \\ (H_s^*(x|\mu_i, \mu_{-i}), L_s^*(x|\mu_i, \mu_{-i})), & \text{if } \mu_i > \mu_{-i} \end{cases}$$

4. for every $x_1^i \in X_1^{\theta,i}$, $\theta = \ell, h$, player $i$ chooses an

$$x_1^i \in \arg\max_{x^i} E[\pi(x_1^i, x_2^i(\mu(x_1^i), \mu(x_1^{-i}), a_\theta), a_\theta)] \equiv BR_i(a_\theta).$$

# Countervailing Incentives

Proof: Given $(\mu_i, \mu_j)$, expected payoffs for each player of each ability are given by

$$v_i(\mu_i, \mu_j, 1) = \begin{cases} \mu_i - \mu_j - \left[ c\left(\frac{c^{-1}(1-\mu_j)}{a_h}\right) - c\left(\frac{c^{-1}(1-\mu_i)}{a_h}\right)\right], & \mu_i \geq \mu_j \\ 0, & \text{otherwise} \end{cases}$$

$$v_i(\mu_i, \mu_j, a_h) = 1 - \min\{\mu_i, \mu_j\} - c\left(\frac{c^{-1}(1 - \min\{\mu_i, \mu_j\})}{a_h}\right).$$

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$$\frac{\partial}{\partial \mu_i} E[v_i(\mu_i, \mu_j, a_h)|\mu_i] = d(\mu_i)(F_{\mu_j}(\mu_i) - 1) < 0$$

$$\frac{\partial}{\partial \mu_i} E[v_i(\mu_i, \mu_j, 1)|\mu_i] = d(\mu_i)F_{\mu_j}(\mu_i) > 0$$

where $d(\mu_i) \equiv \left[1 + \frac{\partial}{\partial \mu_i} c\left(\frac{c^{-1}(1-\mu_i)}{a_h}\right)\right]$ and $F_{\mu_{-i}}(\mu_i) = \Pr(\mu_{-i} \leq \mu_i)$.

$c'(e) > 0$ and $c''(e) \geq 0$, imply $d(\mu_i) \in \left[\frac{a_h - 1}{a_h}, 1\right)$. Back

# Reduced Output

Proof: Ex-ante payoffs for the players are $\frac{K}{2}$ in the two period game, and $\frac{1}{2A}$ in the benchmark, with $\frac{K}{2} > \frac{1}{2A}$.
Ex-ante symmetric implies each will win each game with one half chance in both the repeated contest and single contest.

$$E[\pi_1 + \pi_2] = 1 - E[c(x_1) + c(x_2)] = \frac{K}{2} > \frac{1}{2A} = 1 - E[2c(x)] = E[2\pi_b]$$

This implies that $E[c(x_1) + c(x_2)] < E[2c(x)]$. Also, since $c(\cdot)$ is weakly convex, then

$$E\left[2c\left(\frac{x_1 + x_2}{2}\right)\right] \leq E[c(x_1) + c(x_2)] < E[2c(x)].$$

Because $c(\cdot)$ is strictly increasing, this implies that

$$E[x_1 + x_2] < E[2x].$$

◂ Back