Two-Stage Contests with Private Information

Greg Kubitz

July 12, 2022

Contests

Contests are used to motivate effort by giving a prize to the highest performing contestant

- Firms compete to earn patents or awards
- Employees compete for bonus or promotion
- All-pay auctions

Contestants may have information about their own ability to win the contest

- R&D costs
- Worker productivity
- Value

Contestants use information to choose optimal effort

Dynamic Contests

Contests are often dynamic

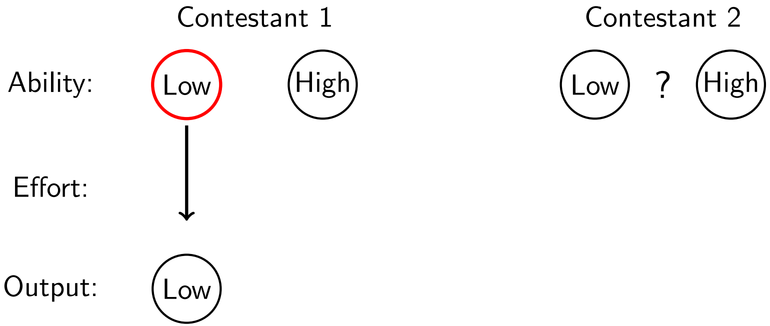
- Contestants compete against one another over several stages

Effort choices can reveal information about ability

- Information can be used later in contest
- How does this affect effort choices and outcomes of contest?
- Do contestants want to reveal ability?

Motivating Example

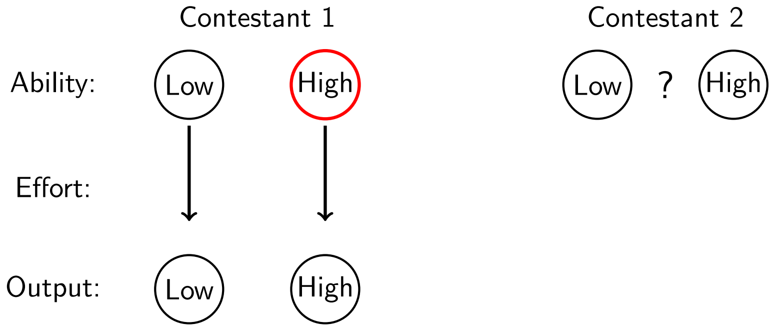
Single contest, opponents ability unknown.



- If Contestant 1 has low ability, chooses low effort and produces low output.

Motivating Example

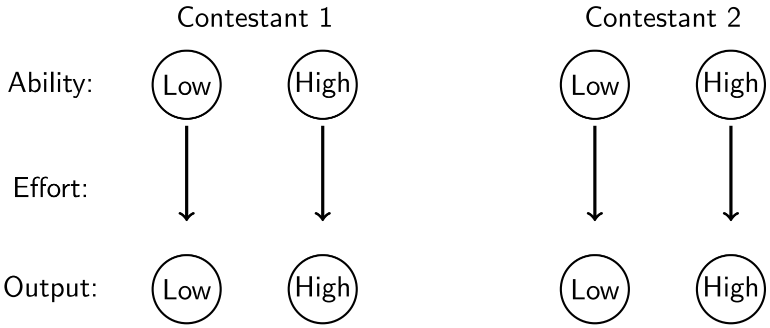
Single contest, opponents ability unknown.



- If Contestant 1 has low ability, chooses low effort and produces low output.
- If Contestant 1 has high ability, produces high output.

Motivating Example

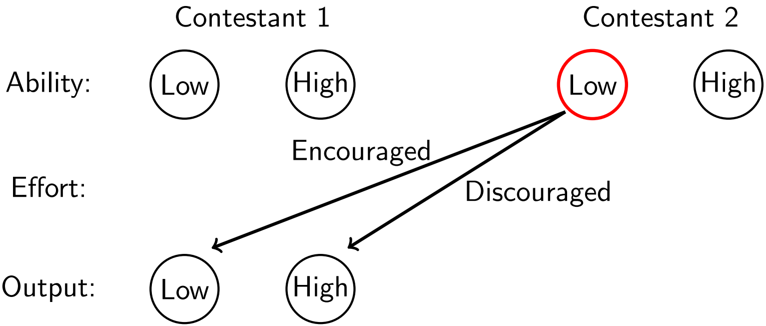
Single contest, opponents ability unknown.



- If Contestant 1 has low ability, chooses low effort and produces low output.
- If Contestant 1 has high ability, produces high output.
- High ability contestant likely to win against low ability one.

Motivating Example

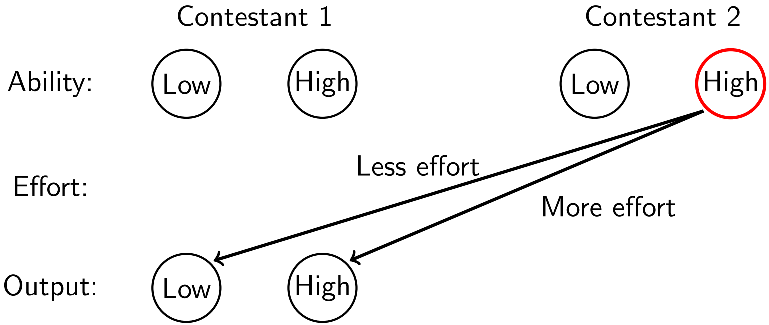
How would Contestant 2 respond to information from first contest?



- If Contestant 2 has low ability, then they are encouraged if they believe Contestant 1 also has low ability, but discouraged if they believe Contestant 1 has high ability.

Motivating Example

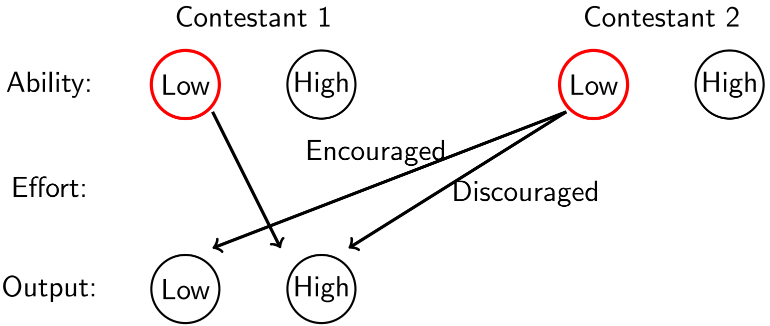
How would Contestant 2 respond to information from first contest?



- If Contestant 2 has high ability, then they use less effort if they believe Contestant 1 also has low ability and use more effort if they believe Contestant 1 has high ability.

Motivating Example

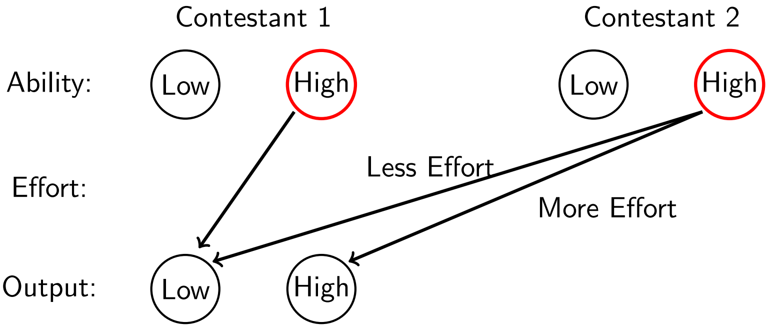
Strategic effects of a second contest



- *Bluffing* - Contestant with low ability wants to appear strong.

Motivating Example

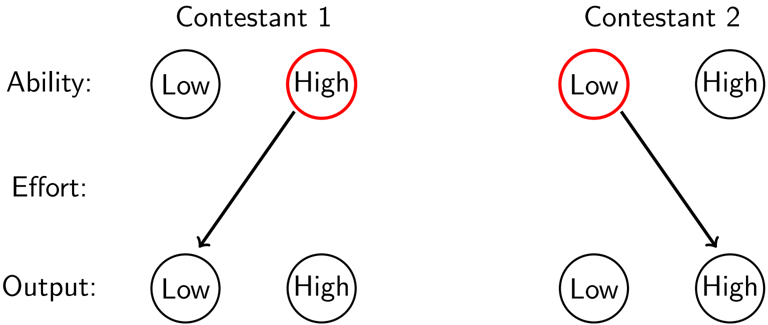
Strategic effects of a second contest



- *Sandbagging* - Contestant with high ability wants to appear weak.

Motivating Example

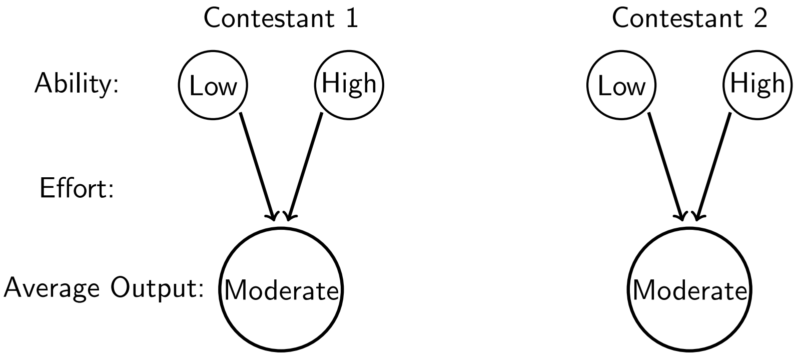
Strategic effects of a second contest



- *Bluffing* - Contestant with low ability wants to appear strong.
- *Sandbagging* - Contestant with high ability wants to appear weak.

Motivating Example

Strategic effects of a second contest



- Unique symmetric equilibrium with partial pooling: effort choice combines countervailing incentives from first and second contest.

Selected Literature

Information manipulation in dynamic competitions:

- Contests - Rosen (1986), Hörner and Sahuguet (2007), Münster (2009), Denter et al. (2021)
- First price auctions - Avery (1998), Ortega Reichert (2001), Bergemann and Hörner (2018)

Information design

- Contestant Types - Zhang and Zhou (2016), Chen et al. (2017), Zheng et al. (2018), Lu et al. (2018), Serena (2021)
- Past outcomes - Wang and Zhang (2009), Aoyagi (2010), Ederer (2010), Klein and Schmutzler (2017), Breig and Kubitz (2021)

Single contest framework - Siegel (2014)

Two-Stage Contest Model

- Two contestants, $i = 1, 2$, independently endowed with ability, $a_i \in \{a^\ell, a^h\}$
  - Normalize $a^\ell = 1$, so a^h ability ratio
  - Ability is privately known, symmetric common prior $Pr(a_i = a^h) = \hat{\mu}$
- In contest t, cost of effort, e_{it}, produces deterministic output $x_{it} = a_i e_{it}$
  - Contestants can "choose" output through effort choice
  - First stage outputs publicly observed before second stage
  - Public history, η_t
- Strategy $\sigma_i = \{H_{it}(x|\eta_t), L_{it}(x|\eta_t)\}_{t=1,2}$, history dependent output distributions for high and low ability contestants
  - Expected output distribution, $F_{it}(x|\eta_t) = \mu_i H_{it}(x|\eta_t) + (1 - \mu_i) L_{it}(x|\eta_t)$

Two-Stage Contest Model

Contestant i's expected payoff in contest t:

$$\mathbb{E}[\pi_{it}(x_{it})|a_i,\sigma_{-i},\eta_t] = p_t\mathbb{E}[w_i(x_{it},x_{-it})|\sigma_{-i},\eta_t] - \frac{x_{it}}{a_i},$$

$$\text{where} \quad w_i(x_{it},x_{-it}) = \begin{cases} 1, & x_{it} > x_{-it} \\ 0, & x_{it} < x_{-it} \\ 1/2, & x_{it} = x_{-it} \end{cases}.$$

Contestants maximize the sum of their payoffs over the two contests

- Prizes for winning contest one, p_1, and contest two p_2, same for each contestant
- Do not discount future payoffs
- No direct discouragement effect

Timeline - Two-stage Contest

Timeline:

1. Symmetric, common priors about ability, $\mu_1 = \mu_2 = \hat{\mu}$
2. Fully persistent abilities privately realized: (a_1, a_2)
3. First stage contest is played
4. Beliefs about opponent's ability updated: $(\mu(x_{11}), \mu(x_{12}))$
5. Second stage contest is played

Symmetric Perfect Bayesian Equilibrium

A set of strategies (σ_i, σ_{-i}) and belief functions $(\mu_i(x), \mu_{-i}(x))$ form a SPBE if

1. output distributions are symmetric for each ability type and history, and belief functions are symmetric: $H_{it}(x|\eta_t) = H_{-it}(x|\eta_t)$, $L_{it}(x|\eta_t) = L_{-it}(x|\eta_t)$, and $\mu_i(x) = \mu_{-i}(x)$,
2. players update beliefs according to Bayes' rule when feasible:

$$\mu_i(x) = \frac{\hat{\mu} h_{i1}(x)}{\hat{\mu} h_{i1}(x) + (1 - \hat{\mu})\ell_{i1}(x)},$$

3. for each ability type and history, the support of the output density is a subset of the associated best response set, $BR^{\theta}_{it}(\sigma_{-i}, \eta_t)$

Second Contest

Characterize equilibrium of two-stage contest by backwards induction

- In the second contest, denote $i = s, w$, where $\mu_s(x_{s1}) \geq \mu_w(x_{w2})$.
- Strong contestant is more likely to be high ability than the weak contestant

Second Contest Equilibrium

Unique Equilibrium (Siegel (2014), Proposition 1, Lemma 1 and 2)

There is a unique Bayesian Nash equilibrium in mixed strategies where, for each player,

1. the output distributions are continuous (except maybe at zero)
2. best response sets are disjoint, monotonic, and have no gaps.

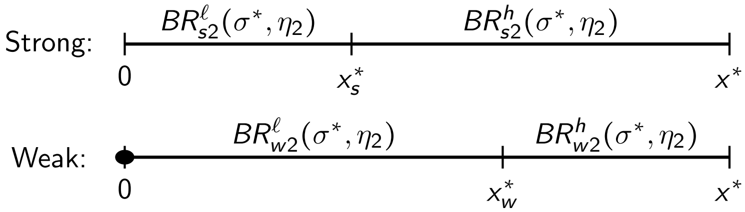


Figure: Best response sets of the strong and weak players in the second contest.

Second Contest Continuation Values

Given beliefs, $\mu_s \geq \mu_w$, the second-stage contest continuation value of contestant i with ability a^θ, denoted $v_i^\theta(\mu_s, \mu_w)$, are

	High Ability	Low Ability
Strong	$p_2\left(\frac{a^h-1}{a^h}\right)(1-\mu_w)$	$p_2\left(\frac{a^h-1}{a^h}\right)(\mu_s-\mu_w)$
Weak	$p_2\left(\frac{a^h-1}{a^h}\right)(1-\mu_w)$	0

Second Contest Continuation Values

Given beliefs, $\mu_s \geq \mu_w$, the second-stage contest continuation value of contestant i with ability a^θ, denoted $v_i^\theta(\mu_s, \mu_w)$, are

	High Ability	Low Ability
Strong	$p_2\left(\frac{a^h-1}{a^h}\right)(1-\mu_w)$	$p_2\left(\frac{a^h-1}{a^h}\right)(\mu_s-\mu_w)$
Weak	$p_2\left(\frac{a^h-1}{a^h}\right)(1-\mu_w)$	0

- Low ability contestant expects positive payoff only when strong contestant
- High ability contestant expects same payoff as strong or weak contestant
  - Increases with the probability that weaker contestant has low ability.

Signaling Incentives

Proposition 1

Expected payoffs in the second contest decrease as μ_i increases for high ability players and increase with μ_i for low ability players.

The marginal effect of beliefs on expected payoffs in the second contest

$$\frac{\partial}{\partial \mu_i}\mathbb{E}[v_i^h(\mu_i, \mu_{-i})] = -p_2\left(\frac{a^h - 1}{a^h}\right)(1 - F_{\mu_{-i}}(\mu_i)) < 0$$

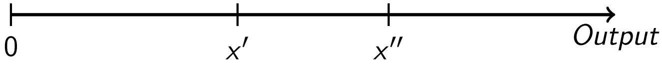
$$\frac{\partial}{\partial \mu_i}\mathbb{E}[v_i^\ell(\mu_i, \mu_{-i})] = p_2\left(\frac{a^h - 1}{a^h}\right)F_{\mu_{-i}}(\mu_i) > 0$$

where $F_{\mu_{-i}}(M) = \Pr(\mu_{-i} \leq M)$

Monotonic Beliefs

Lemma

In every SPBE, $\mu^*(x)$ is weakly increasing in output over the support of first period strategies.

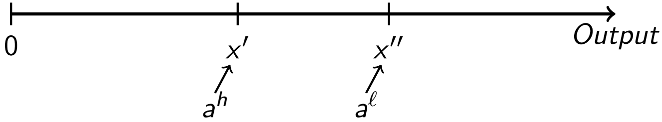


$$0 \le \mu(x'') < \mu(x') \le 1$$

Monotonic Beliefs

Lemma

In every SPBE, $\mu^*(x)$ is weakly increasing in output over the support of first period strategies.



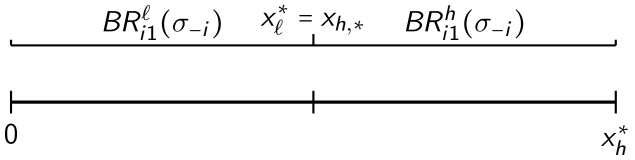
$$0 \leq \mu(x'') < \mu(x') \leq 1$$

- Current marginal costs (as in single contest)
- Value of the second contest (countervailing incentives)

Best Response Sets are Overlapping Intervals

Lemma

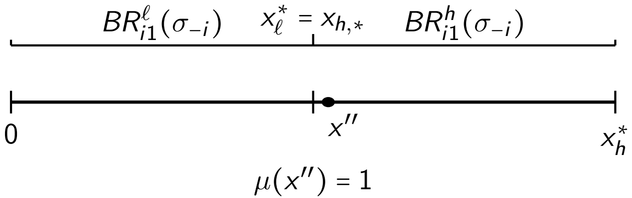
$BR^{\ell}_{i1}(\sigma^*)$ and $BR^{h}_{i1}(\sigma^*)$ are intervals where $0 = x_{\ell,*} \leq x_{h,*} < x^*_{\ell} \leq x^*_h$.



Best Response Sets are Overlapping Intervals

Lemma

$BR^{\ell}_{i1}(\sigma^*)$ and $BR^{h}_{i1}(\sigma^*)$ are intervals where $0 = x_{\ell,*} \leq x_{h,*} < x^*_{\ell} \leq x^*_h$.

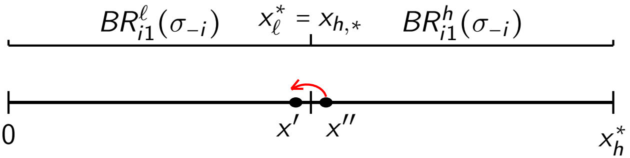


Payoffs continuous in first contest output, jump in expected payoffs in second contest

Best Response Sets are Overlapping Intervals

Lemma

$BR^{\ell}_{i1}(\sigma^*)$ and $BR^{h}_{i1}(\sigma^*)$ are intervals where $0 = x_{\ell,*} \leq x_{h,*} < x^*_{\ell} \leq x^*_{h}$.



$$\mu(x'') = 1$$

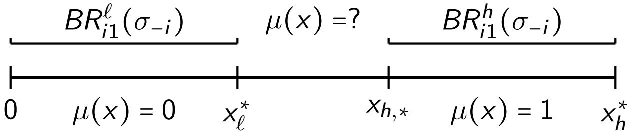
$$\mu(x') = 0$$

Payoffs continuous in first contest output, jump in expected payoffs in second contest

Best Response Sets are Overlapping Intervals

Lemma

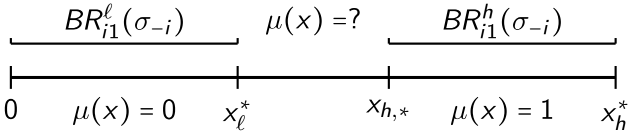
$BR^{\ell}_{i1}(\sigma^*)$ and $BR^{h}_{i1}(\sigma^*)$ are intervals where $0 = x_{\ell,*} \leq x_{h,*} < x^*_{\ell} \leq x^*_h$.



Best Response Sets are Overlapping Intervals

Lemma

$BR^{\ell}_{i1}(\sigma^*)$ and $BR^{h}_{i1}(\sigma^*)$ are intervals where $0 = x_{\ell,*} \leq x_{h,*} < x^*_{\ell} \leq x^*_h$.



- If $\mu(x) \approx 1$, then low ability players prefer outputs off equilibrium path
- If $\mu(x) \approx 0$, then high ability players prefer outputs off equilibrium path

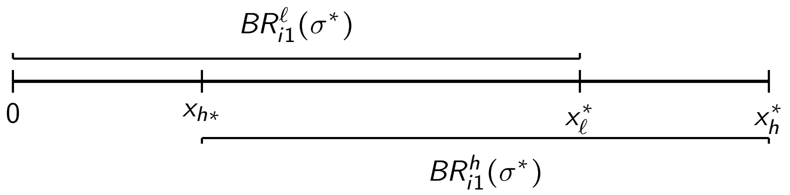
No beliefs off equilibrium path that punish both high and low ability contestants.

Two-Stage Contest Equilibrium

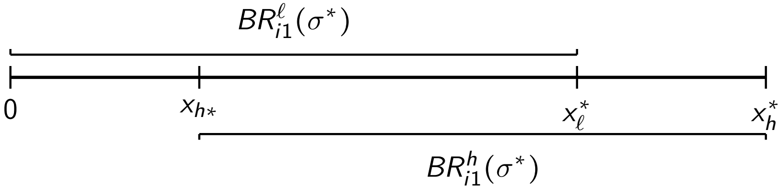
Theorem 1

There is a unique SPBE in the two-stage contest where in the first contest

1. the output distributions are continuous,
2. the belief function is continuous and weakly increasing in output, and
3. best response sets for high and low ability contestants are intervals with non-trivial overlap.



Equilibrium Conditions



On intersection of best response sets, $[x_{h*}, x^*_{\ell}]$, indifference conditions require

$$\mu^*(x) = \frac{x - x_{h*}}{p_2} + \mu^*(x_{h*})$$

$$F^*_1(x) = \frac{a^h}{a^h - 1} - \left(\frac{a^h}{a^h - 1} - \frac{x_{h*}}{p_1}\right) e^{-\frac{a^h - 1}{a^h p_1}(x - x_{h*})}$$

Welfare

Expected payoffs in two-stage contest:

$$K^h(p_1, p_2) = \frac{(a^h - 1)\left(x_{h*} + p_2\left(1 - \mu^*(x_{h*})\right)\right)}{a^h}$$

$$K^\ell(p_1, p_2) = 0$$

Ex-ante expected output of each contestant in two-stage contest:

$$Y(p_1, p_2) = \frac{p_1 + p_2}{2} + (a^h - 1)\hat{\mu} R^h(p_1, p_2) - a^h \hat{\mu} K^h(p_1, p_2)$$

- $R^h(p_1, p_2)$ is expected revenue of high ability contestant
- Output increases as payoffs decrease and high ability contestant revenue increases

Prize Structure and Surprise Victories

Relative prizes in two contest impact amount of pooling in the first contest, and therefore information available in second contest

Proposition 3

A higher prize ratio, p_1/p_2

1. decreases the probability of that a low ability player wins the first contest against a high ability player
2. increases this probability in the second contest

- First contest: Incentive to separate becomes stronger relative to incentive to pool
- Second contest: Beliefs about contestants' abilities become more asymmetric, on average

Output

Proposition 4

When $\hat{\mu} \geq 1/2$ and $p_1 + p_2 = \bar{p}$, expected total output over the two contests is maximized and expected payoffs for high ability contestants are minimized when either $p_1 = \bar{p}$ and $p_2 = 0$ or $p_1 = 0$ and $p_2 = \bar{p}$.

With positive prizes in both stages

- Sandbagging and bluffing have opposite impacts in the first stage; reduction in effort from high ability contestants has larger impact
- Partial pooling typically leads to asymmetric beliefs and less competitive second stage
- Surprise victories occur with positive probability

Prize Structure: Payoffs and Output

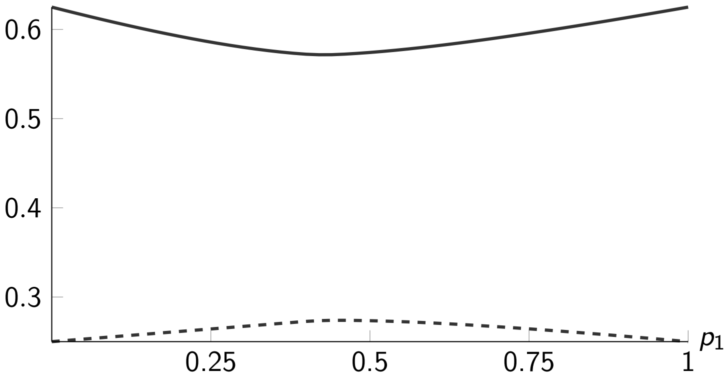


Figure: Expected payoffs of the high ability contestant (dashed) and ex-ante expected output of each contestant (solid) for different prize ratios when $p_1 + p_2 = 1$, $\hat{\mu} = 1/2$ and $a^h = 2$.

Convex Cost of Effort

Theorem 2

Let $p_1, p_2 > 0$ and let $c(e)$ be twice differentiable on the non-negative reals, strictly increasing and weakly convex, with the cost of zero effort being zero. There is a unique SPBE of the two-stage contest.

Welfare

- With convex costs, lower output when almost all stakes are placed on one contest
- Evenly split prizes over two contests tend to increase expected payoffs and reduce expected output
  - Small prize in first contest - pooling maintains competitive second contest
  - Small prize in second contest - less competitive second contest has small impact

Prize Structure: Payoffs and Output - Convex Cost

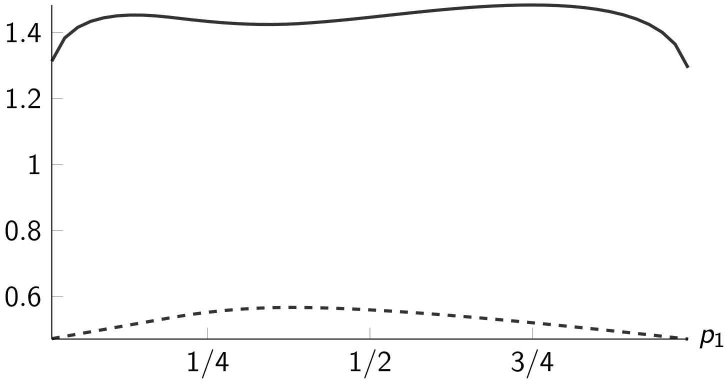


Figure: Expected payoffs of the high ability contestant (dashed) and ex-ante expected output (solid) of each contestant for different prize ratios when $p_1 + p_2 = 1$, $\hat{\mu} = 1/2$ and $a^h = 2$, $c(e) = e^4$.

Conclusion

"Appear weak when you are strong, and strong when you are weak" -Sun Tzu

In two-stage contests sandbagging and bluffing are the result of countervailing incentives from future competition

- Partial pooling equilibrium in first-stage contest where higher output leads to a (weakly) higher belief about contestant's ability
- Surprise victories possible in both stages

Prize allocation to maximize output

- Prize all in one contest with linear costs and contestants likely to be high ability
- With convex costs, uneven prizes mitigates output reduction from two-stage contests