Industry Costs and Research Aggregation in Dynamic Competition

Greg Kubitz* and Kyle Woodward†

December 2018

Abstract

We study information acquisition and subsequent price competition in an environment where the cost of each firm is initially unknown and composed of two components, private costs specific to the firm and costs common to all firms in the industry. In this setting, firms choose high initial prices to soften future competition. Moreover, this pricing distortion is exacerbated when firms only possess private information about firm specific costs. This implies that sharing information about industry relevant costs, such as aggregating cost information through a trade association, will lead to higher prices. Additionally, when firms share information about common costs they have less incentive to acquire information about firm specific costs.

1 Introduction

Firms within an industry often share information through a trade association. Examples include information about market demand, firm costs, capacity, pricing and sales. Antitrust considerations of collusive potential and the uniformity effect have created clear guidelines for sharing prices and, to a lesser extent, capacity and sales. The impacts, and therefore the guidelines, of sharing cost information through a trade association are not as clear.[1]

In this paper we consider the impact of sharing industry relevant cost information via a trade association in the context of dynamic price competition. Information that is not shared through the trade association may otherwise be endogenously revealed through observed prices. The presence of this information exchange agreement can change pricing behavior

*QUT; gregory.kubitz@qut.edu.au

†UNC–CH; kyle.woodward@unc.edu

[1]See discussions in Kühn and Vives [1994] and Kühn [2001].

directly, as there is more information available to the firm, and indirectly as the information content of prices also changes. We additionally analyze the impact of an agreement on incentives to acquire information prior to the exchange. Overall, we show that sharing cost information via a trade association can lead to higher prices and less acquisition of information prior to exchange.

We study a stylized dynamic Bertrand competition model where firms face a commonly known demand curve and have private information about costs. The structure of costs is generalized to allow for both idiosyncratic costs that are firm specific and costs that are common to all firms in the industry, e.g. prices of inputs. In this competitive setting, firms have an incentive to over represent costs by setting a higher price. Understanding that competing firms have other channels to learn about common costs, this signal jamming focuses on inflating beliefs about firm specific costs. The signal is confounded however as a high price could be the result of the firm having information about high costs that are either firm specific or affect all firms in the industry.

Sharing common cost information changes each firm's absolute and relative information structure. Identical information on shared costs implies that unexpectedly high or low prices of another firm is reflective of the specific firm's cost structure. With prices acting as a clearer signal, the incentive to signal jam strengthens, resulting in higher average prices. Additionally, prices are more informative about the firm's idiosyncratic cost structure. This reduces the informational rents of firm-specific cost information lowering firms' willingness to pay to acquire it.

Our analysis begins with two stage price competition between two firms. Each firm's marginal cost has two independent components; one is firm specific and the other, called the common component, is shared with the other firm. The true value of the three total cost components are initially unknown and distributed independently. Each firm has conditionally independent private signals over each of their own components. In the first stage, prices are chosen based on these signals. At the end of this stage, firms obtain full information of their own cost structure, both individual and common components. Additionally, they observe the price chosen by the other firm. Given this information, firms choose prices in the second stage of competition.

For any symmetric level of signal precision we show that the dynamic pricing game has a unique symmetric equilibrium in linear strategies. A complication arises from the fact that price acts as a signal for two sources of private information, only one of which, the firm specific costs, is relevant to the other firm. Prices therefore are informative about private cost information but linear pricing strategies are not separating in the sense of revealing this fully. The optimal first stage pricing strategy depends on the amount of information the price conveys to the other firm about this cost component. In equilibrium, the informativeness of the price necessarily depends on the pricing strategy. Identifying equilibrium requires finding the pricing strategy that leads to a level of price informativeness for which the pricing strategy is optimal.

A linear pricing strategy is characterized by coefficients which determine how much the firm's price changes with the expected cost of each component. In any symmetric linear equilibrium, prices in the first period vary less with firm specific costs than is optimal in the one-shot pricing game. Moreover, the equilibrium value of this coefficient is inversely proportional to the informativeness of the price in equilibrium. This reflects the firm's incentive to signal jam in order to maintain informational rents in the second period.

Equilibrium pricing coefficients also depend on the precision of signals received by firms prior to competition. More precise signals of firm-specific costs increases the incentive to signal jam and leads the firm reduce the pricing coefficient of this information. Conversely, intermediate levels of precision on the common cost component confound the price signal allowing firms to use more information about firm specific costs without losing as much informational rents. Therefore the relationship between the equilibrium pricing coefficient on firm specific cost and common cost signal precision is hump-shaped.

Within this framework, we analyze the impact of firms sharing signals on the common cost component in a verifiable way. The only source of private information each firm still posses is a private cost signal. Therefore, in equilibrium, first stage prices will be a better signal of this information despite a smaller equilibrium pricing coefficient. This increases the incentive to signal jam in the first stage leading to higher prices on average. Signal jamming also distorts the equilibrium pricing strategy away from the optimal one-shot strategy and reduces how much the price adjusts to firm specific expected costs. This lowers the value of this information to the firm, and therefore the willingness to pay for a more precise signal prior to price competition.

This paper adds to the literature of dynamic oligopoly models with incomplete information and perfect monitoring that examine the information content of prices and the resulting pricing distortions that stem from signaling. Mailath [1989] and Mester [1992] identify the incentive to over represent cost by choosing a high price and more recently Jeitschko et al. [2018] place this incentive in the context of information sharing and information acquisition. Our paper allows for multiple sources of private information allowing for a richer information set and a more complex strategy space. Specifically it allows for firms to posses information that is relevant to their competitors for more than strategic reasons.

As equilibrium strategies do not fully reveal private information this paper also adds to the literature of signal jamming literature in dynamic oligopoly models. In a Cournot setting where market price is observed, Bonatti et al. [2017] characterize with a continuous time model the dynamics of simultaneous signal jamming and learning when firms begin with private cost information. Mirman et al. [1993] look at the case where firms have private information about individual demand curves. In our setting of perfect monitoring, the signal jamming comes from how much weight the firm places on each source of information when choosing price. By reducing the weight on one source of information the firm reduces the informativeness of the price about this source.[2]

Our paper also adds to the literature on optimal use of information from private and public signals. When public signals are perfectly correlated, Angeletos and Pavan [2007] identify both the equilibrium and efficient uses of information in a flexible setting that assumes quadratic payoffs and Gaussian information. They show that when actions are strategic complements agents weight public information relatively more, while the opposite is true under strategic substitutes. While the current paper has this information structure when firms share common cost information, we show these properties hold even when the common costs signals are correlated, but not shared.

When public signals are perfectly correlated, Bergemann and Morris [2013] show firms prefer to share an intermediate level of information about demand uncertainty.[3] When public signals are allowed to be partially correlated Bergemann et al. [2018] show that informational intermediaries may have an incentive to garble private (idiosyncratic) information about consumers before sharing with a firm. This lower the rents needed to pay the consumer for their information. While intermediate levels of information transmission are not allowed in the decision to share information through the trade association, the pricing strategy in the first stage of competition determines the amount of information revealed to the other firm in the case where the trade association is not used. This offers an alternative mechanism to partial revelation that emerges through dynamic competition.

Lastly our paper is related to the literature of information acquisition from multiple heterogeneous sources. We show that the promise of a more precise public signal crowds out the willingness to pay for a more precise private signal. This is consistent with the equilibrium result of Colombo et al. [2014]. Additional work on the type of information that is preferred include Morris and Shin [2002] and Myatt and Wallace [2011] in the setting of a generalized beauty contest as well as Myatt and Wallace [2015a] and Myatt and Wallace [2015b] in oligopoly settings with uncertainty on demand.

The rest of the paper is organized as follows. Sections 2 and 3 respectively introduce

[2]Mirman et al. [1994] identify that an increase in the choice of quantity will reduce the informativeness of price in the Cournot setting where incomplete information is symmetric.

[3]Sharing all or no information though a intermediary such as a trade association has been extensively studied in the case of static oligopoly competition. See Raith [1996] and Vives [2001].

and analyze the model of two stage price competition with and without information sharing. Section 4 introduces information acquisition prior to the two stages of competition. Section 5 concludes.

2 Model

Two firms, i and j, compete for market share over two periods $t = 1, 2$. Demand is linear in prices, symmetric across firms, and time-independent. Firm i's demand is given by

$$q_{i,t} = a - bp_{i,t} + ep_{j,t}.$$[4]

We assume that demand is weakly more sensitive to a firm's own price than to its opponent's, so that $|e| \leq b$. Each firm faces a constant marginal cost c_i that is same in each period, so profits are

$$\pi_{i,t} = (p_{i,t} - c_i)\, q_{i,t}.$$

Firms are initially uncertain about their marginal costs of production, but know that costs are comprised of an idiosyncratic component θ_i and a common component ρ; their constant marginal cost is the sum of the two components, $c_i = \rho + \theta_i$. We assume that cost components are joint-normally distibuted with zero covariance, so that

$$\begin{pmatrix} \theta_i \\ \theta_j \\ \rho \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_\theta \\ \mu_\theta \\ \mu_\rho \end{pmatrix}, \begin{pmatrix} \sigma_\theta^2 & 0 & 0 \\ 0 & \sigma_\theta^2 & 0 \\ 0 & 0 & \sigma_\rho^2 \end{pmatrix}\right).$$

Throughout we will denote the precision of the random variable x by $\tau_x = 1/\sigma_x^2$.

Play proceeds in two stages. In the first stage, each firm receives two noisy signals, $s_{i,\theta}$ and $s_{i,\rho}$, of the values of their idiosyncratic and common costs, respectively. These signals are normally distributed with uncorrelated error terms, and the error terms are uncorrelated between firms. We model these signals as $s_{i,x} = x + \varepsilon_{i,x}$, where $\varepsilon_{i,x}$ is normally distributed with variance $\sigma_{s,x,i}^2$.[5] Upon the realization of their private signals, firms simultaneously select prices $p_{i,1}$ and obtain stage profits $\pi_{i,1}$.

After first-stage profits are obtained, firms become perfectly informed of both the common and their (individual) idiosyncratic cost components. Each also witnesses its opponent's first-

[4]Unless otherwise specified, our equations and inequalities should be taken to be symmetric for agent j.

[5]For the majority of our results we will assume symmetry, so that $\sigma_{s,x,i}^2 = \sigma_{s,x,j}^2$. Allowing for heterogeneity in the variance of the error term is essential when we discuss information acquisition.

stage price, but remains unaware of its opponent's idiosyncratic cost component.[6] Firms then compete again by simultaneously selecting prices and obtain stage profits $\pi_{i,2}$.

The game ends after the second stage, and ex post utility is the (undiscounted) sum of stage profits,

$$u_i(p_i, p_j) = \pi_{i,1}(p_{i,1}, p_{j,1}) + \pi_{i,t}(p_{i,2}, p_{j,2}).$$

We restrict attention to *subgame perfect equilibria in linear strategies.*

Analysis proceeds in three parts. We first determine properties of equilibrium in this base model. Then, we allow firms to share information about their common cost component prior to the two-stage competition. Lastly, we allow firms to acquire more precise information about their costs, and compare the amount of information acquired in the setting where firms share common cost information to the setting where they do not.

3 Equilibrium

We compute the pricing equilibrium in the two stage model by backwards induction. In a subgame-perfect equilibrium second-period prices are best responses to available information. However, even in an equilibrium where first period prices are strictly monotone in each signal, it is impossible for private information to be fully-revealed as in a standard separating equilibrium as private information is two-dimensional while actions are one-dimensional and monotone in information. Residual uncertainty in the second stage is an important feature in our model, affecting firms' first-period pricing through their ability to distort publicly-available information about their costs.

3.1 Second period pricing

In the second period, each firm knows its own marginal costs precisely, but knows only the distribution over its opponent's costs. Letting $F^j(\cdot; p_{i,1}, p_{j,1}, \rho) \equiv F^j$ be the distribution of firm j's second period price conditional on firm i's available information,[7] the profit maximization problem is

$$\max_p \int (p - c_i)(a - bp + ex)\, dF^j(x).$$

[6]Since demand is a deterministic function of firm prices, the assumption that firms witness each others' prices is sufficient to imply that they are perfectly informed of their own private cost c_i; alternatively, if they witness their own sales volume they will be perfectly aware of their opponent's price. That they obtain perfect knowledge of each of the components of $c_i = \rho + \theta_i$ is an additional assumption.

[7]Firm i also knows θ_i, $s_{i,\theta}$, and $s_{i,\rho}$, but these offer no payoff-relevant information in the second stage (beyond θ_i, ρ, and $p_{i,1}$) and may be ignored.

Lemma 1. *Firm i's optimal second period price is*

$$p^\star_{i,2} = \frac{1}{2b}\left(a + bc_i + e\mathbb{E}\left[\left. p^\star_{j,2}\right| \rho, p_{i,1}, p_{j,1}\right]\right).$$

Firm i's maximum second period expected profit is

$$\pi^\star_{i,2} = \frac{1}{4b}\left(a - bc_i + e\mathbb{E}\left[\left. p^\star_{j,2}\right| \rho, p_{i,1}, p_{j,1}\right]\right)^2.$$

Thus firm i's second period price is an affine function of the demand intercept, its (known) cost $c_i = \rho + \theta_i$, and its expectation over firm j's second period price. Profits then have a standard quadratic form.

Lemma 2. *In any equilibrium, expected second period prices of a firm given publically available information are*

$$\mathbb{E}\left[\left. p^\star_{j,2}\right| \rho, p_{i,1}, p_{j,1}\right] = \frac{1}{4b^2 - e^2}\left(\left(2b+e\right)a + 2b^2\mathbb{E}\left[\left. c_j\right| \rho, p_{j,1}\right] + be\mathbb{E}\left[\left. c_i\right| \rho, p_{i,1}\right]\right),$$

which result in the following expected second period profits:

$$\pi^\star_{i,2} = \frac{1}{4b}\left(\frac{1}{4b^2 - e^2}\right)^2\left(\left(4b^2+2be\right)a - 4b^3c_i + \left(\mathbb{E}\left[\left. c_i\right| \rho, p_{i,1}\right] - c_i\right)be^2 + 2b^2e\mathbb{E}\left[\left. c_j\right| \rho, p_{j,1}\right]\right)^2.$$

Lemma 2 connects firms i's expected second period profits to its first period price. These profits increase in $E[c_i|\rho, p_{i,1}]$, the expectation of firm i's cost given information available to firm j in the second period. Therefore firm i has an incentive to over-represent its cost, leading firm j to increase its second period price, softening competition for firm i.[8]

3.2 First period pricing

First period prices are set to optimize the sum of profits over two periods. Although first period prices have no direct effect on second period profits, firm i's price affects firm j's beliefs regarding firm i's costs. This is shown directly in Lemma 2, where $p_{i,1}$ enters only in $\mathbb{E}[c_i|\rho, p_{i,1}]$.

Firm i's first period profit maximization problem is

$$\max_p \mathbb{E}\left[\left.\pi_{i,1}\right| s_{i,\rho}, s_{i,\theta}\right] + \mathbb{E}\left[\left.\pi^\star_{i,2}\right| s_{i,\rho}, s_{i,\theta}\right] = \max_p \mathbb{E}\left[\left.\left(a - bp + e\hat{p}_{j,1}\right)\left(p - c_i\right) + \pi^\star_{i,2}\right| s_{i,\rho}, s_{i,\theta}\right].$$

[8]This describes the reaction of firm j when the firms are selling substitutes, $e > 0$. When $e < 0$, a higher value of $E[c_i|\rho, p_{i,1}]$ leads to a lower $p^*_{j,2}$ which still increases $\pi^*_{i,2}$. When $e = 0$, the price and profit equations reduce to the standard monopoly model.

A marginal increase in first period price affects first period profits in a standard way, and has an additional effect on second period profits by manipulation of the opposing firm's second period beliefs which changes second period price choices. This gives Lemma 3.

Lemma 3. *Optimal first period prices are given by*

$$p^{\star}_{i,1} = \left(\frac{1}{2b}\right)\mathbb{E}\left[bc_i + a + e\hat{p}_{j,1}\middle|\, s_{i,\rho}, s_{i,\theta}\right]$$
$$+ e\left(\frac{1}{2b}\right)^2 \mathbb{E}\left[\left(a - bc_i + e\mathbb{E}\left[p^{\star}_{j,2}\middle|\, \rho, p^{\star}_{i,1}, \hat{p}_{j,1}\right]\right)\frac{\partial}{\partial p_{i,1}}\mathbb{E}\left[p^{\star}_{j,2}\middle|\, \rho, p^{\star}_{i,1}, \hat{p}_{j,1}\right]\middle|\, s_{i,\rho}, s_{i,\theta}\right].$$

We constrain attention to equilibria in pricing strategies that are linear in the expected value of each cost component.[9] A linear first period price can be expressed as

$$p_{i,1} = p_{i,0} + \mathbb{E}\left[\theta_i\middle|\, s_{i,\theta}\right]p_{i,\theta} + \mathbb{E}\left[\rho\middle|\, s_{i,\rho}\right]p_{i,\rho}.$$

Under linear strategies, each firm's first period price choice is a normally distributed random variable from the perspective of the other firm. Therefore, $(c_i, \rho, p_{i,1})$ are distributed joint-normally, which implies that $\mathbb{E}[c_i|\rho, p_{i,1}]$ is linear in $p_{i,1}$. Moreover, the effect of an increase in firm i's first period price on firm j's second period beliefs, and hence second period price, is constant and independent of the level of price. Conditioning beliefs on this relationship gives Lemma 4.

Lemma 4. *The marginal effect of firm i's first period price on firm j's expected second period price is*

$$\frac{\partial}{\partial p_{i,1}}\mathbb{E}\left[p^{\star}_{j,2}\middle|\, \rho, p_{i,1}\right] = \frac{be}{4b^2 - e^2}\kappa_i,$$

$$\text{where } \kappa_i \equiv \frac{\partial}{\partial p_{i,1}}\mathbb{E}\left[c_i\middle|\, \rho, p_{i,1}\right] = \frac{\sigma^2_{\theta}\bar{\tau}_{s,\theta,i}p_{i,\theta}}{\sigma^2_{s,\rho}\bar{\tau}^2_{s,\rho,i}p^2_{i,\rho} + \left(\sigma^2_{\theta} + \sigma^2_{s,\theta,i}\right)\bar{\tau}^2_{s,\theta,i}p^2_{i,\theta}} \text{ and } \bar{\tau}_{s,x,i} = \frac{\tau_{s,x,i}}{\tau_x + \tau_{s,x,i}}.$$

The term κ_i captures the relative informativeness of firm i's first period price regarding the its idiosyncratic cost component θ_i, the remaining source of asymmetric information in the second period when ρ is commonly known. Despite observing ρ, firms do not observe each other's first period signal on the common cost component, $s_{i,\rho}$. Because the first period price depends on the realization of $s_{i,\rho}$, it can be thought of a noisy signal of $s_{i,\theta}$, Therefore

[9] This forces each firm to commit to using information about each cost component at a fixed level for all possible signals, $(s_{i,\theta}, s_{i,\rho})$, it may receive. As we show, these strategies are best responses to the opponent's linear pricing rule, even allowing for nonlinear pricing rules, when the firm has received its private signals. It is possible that there exist equilibria in nonlinear pricing rules.

the informativeness of the price in determining θ_i depends not only on the variance of the price relative to $s_{i,\theta}$ but also relative to $s_{i,\rho}$.[10]

For $x \in \{\rho, \theta\}$, $\bar{\tau}_{s,x,i}$ is the relative contribution of the normally distributed noise in firm i's signal around the true parameter x, to the precision of the signal $s_{i,x}$. When signals are very noisy, $\bar{\tau}_{s,x,i}$ will be close to zero; when signals give a more precise prediction of the true cost parameter, $\bar{\tau}_{s,x,i}$ will be close to one. When signals are more precise, they have a larger role in the formation of expectations over the cost parameters.

The term $\bar{\tau}_{s,x,i} p_{i,x}$ is the derivative of first period price with respect to $s_{i,x}$, and affects the informativeness of the first period price about the firms cost. Therefore the choice of strategy in the first period for a given level of information precision will directly impact the value of κ_i. Specifically, as either $p_{i,\theta}$ or $p_{i,\rho}$ increases, κ_i decreases. If a firm increases $p_{i,x}$ while precisions remains constant, it is increasing the variance of price and therefore changes in price will be less informative of the underlying primitives of the model. Moreover, the incentive constraints of the equilibrium strategy in the first period depend on the value of κ_i.[11] This fixed point problem is expressed in the single-variable equation in Proposition 1. Importantly, since pricing strategies are not observed, κ_i is not affected by firm i's selection of price; it is determined by the pricing strategy the firm is believed to be following.

Proposition 1. *There exists a unique symmetric equilibrium in linear pricing strategies. The equilibrium strategies are determined by the value of κ in equilibrium which satisfies the following single variable equation:*

$$\kappa = \frac{\sigma_\theta^2 \bar{\tau}_{s,\theta} p_\theta}{\sigma_{s,\rho}^2 \bar{\tau}_{s,\rho}^2 p_\rho^2 + \left(\sigma_\theta^2 + \sigma_{s,\theta}^2\right) \bar{\tau}_{s,\theta}^2 p_\theta^2},$$

$$\text{subject to } p_\theta = \frac{1}{2 + \beta\kappa} \text{ and } p_\rho = \frac{1 - \left(\frac{b-e}{2b-e}\right)\beta\kappa}{2 - \frac{e}{b}\bar{\tau}_{s,\rho} - \frac{1}{2}\left(1 - \bar{\tau}_{s,\rho}\right)\beta^2\kappa^2},$$

where $\beta = \frac{e^2}{4b^2 - e^2}$.

There are two strategic effects we can identify in the first period prices. First, due to the correlation of one cost signal and the independence of the other signal, firms may want to act more heavily on one of these signals than the other if they prefer to have their prices correlated in the first period. Additionally, firms benefit from having private information in the second period and therefore prefer to not reveal precise information about their idiosyncratic cost term. The implications of the first effect are in Proposition 2 and those of the second effect are in Proposition 3.

[10] Note that σ_ρ^2 does not appear in the denominator of κ_i since ρ is commonly observed.

[11] Note that a deviation does not indicate a specific misreport of marginal cost but rather an iso-information curve of feasible $s_{i,\rho}, s_{i,\theta}$. These iso-information curves depend on the value of κ_i.

Proposition 2. *In equilibrium, $p_\rho < p_\theta$ when goods are complements ($e < 0$) and $p_\rho > p_\theta$ when goods are substitutes ($e > 0$). $p_\rho = p_\theta$ when markets are independent ($e = 0$).*

When $e > 0$, so that goods are substitutes, firms' first period prices are more sensitive to information on the common cost component than to information on their idiosyncratic cost component. If a firm receives a high signal on the common cost component this often implies the other firm will set a high price, increasing demand and making it optimal to further increase price. When $e < 0$, so that goods are complements, prices are strategic substitutes and will not respond strongly to the common cost signal. When $e = 0$, so that there are no cross-firm demand effects, there is no need to either adjust for the opponent's price and information about each cost component affects first period prices identically. Moreover, in the monopoly case there will be no attempt to conceal information regarding cost. However, in general the information conveyed by first period prices will affect second period profits. Proposition 3 illustrates firms' incentives to not reveal too much information on their idiosyncratic cost component.

Proposition 3. *The equilibrium values of p_θ and κ are inversely related: p_θ increases when κ decreases and vice versa. Additionally, p_θ is decreasing and κ is increasing in $\bar{\tau}_{s,\theta}$, and there is a $\tau^\star$ such that for all $\bar{\tau}_{s,\rho} > \tau^\star$, κ is increasing and p_θ is decreasing in $\bar{\tau}_{s,\rho}$, and for all $\bar{\tau}_{s,\rho} < \tau^\star$, κ is decreasing and p_θ is increasing in $\bar{\tau}_{s,\rho}$. When $e > 0$, $\tau^\star > 1/2$ and when $e < 0$, $\tau^\star < 1/2$.*

When $\bar{\tau}_{s,\theta}$ is close to one, signals relatively precise information about θ_i. To maintain the strategic advantage of private information, the firm will use less of the information from a precise signal when determining first period price. If this signal is not precise, then even if the price fully reflects the information in the signal, it will still maintain private information in the second period from learning the true value of θ_i.

The presence of uncertainty on the common component of cost adds noise to the relationship between first period price and the signal on idiosyncratic cost. When this relationship is more noisy, the price reveals less information about the idiosyncratic signal, allowing the firm to use this information in its pricing decision without revealing too much information. If the signal about the common cost is relatively imprecise, $\bar{\tau}_{s,\rho}$ close to 0, then firms do not learn much information from this signal, and relatively little noise is added to this relationship. Additionally, if the signal is very precise, $\bar{\tau}_{s,\rho}$ close to 1, then when firms learn the true value of ρ in the second round, they will learn, with little error, what signal $s_{i,\rho}$ their opponents received and will be able to tease apart the noise in the pricing strategy. Therefore an intermediate level of precision $\bar{\tau}_{s,\rho}$ on signal $s_{i,\rho}$ will maximize p_θ for a given value of $\bar{\tau}_{s,\theta}$.

In general the incentives to hide idiosyncratic cost information leads firms to be less responsive to their idiosyncratic cost signal than is optimal in a one-stage game (without the informational channels implied by our two-stage model), so relaxing signal jamming incentives leads to an increased sensitivity of price to information on the idiosyncratic cost component.

3.3 Sharing industry relevant information

We now consider the effect of the firms sharing information about costs through a trade association. We assume that signals about the common cost component are shared, while those of firm's idiosyncratic shocks are not. Information shared via a trade association is that which is relevant to the production process of all firms, e.g. input costs, and firms prefer to maintain private information about idiosyncratic costs.

When firms share their signals about their common cost component they will have the same expectation about this parameter. This simplifies the two stage competition model to a generalization of the single cost component model in Mailath [1989] and **?**. While there are still two cost components, the informational structure is simplified so that firms posses private information about their only idiosyncratic cost components; the remaining uncertainty regarding the common cost component is common to both firms. While the optimality conditions look similar in this setting, the equilibrium pricing strategies in the first period fully reveal the private information of each firm. We briefly outline the significant differences from the previous section.[12]

In the second period the information that is available to each firm now includes $s_\rho = (s_{i,\rho}, s_{j,\rho})$. The new first order conditions are given in Lemma 5.

Lemma 5. *Firm i's optimal second period price is*

$$p_{i,2}^c = \frac{1}{2b}\left(a + bc_i + e\mathbb{E}\left[p_{j,2}^c\middle|\,\rho, p_1, s_\rho\right]\right).$$

Firm i's optimal first period price is

$$p_{i,1}^c = \left(\frac{1}{2b}\right)\mathbb{E}\left[bc_i + a + ep_{j,1}^c\middle|\,s_\rho, s_{i,\theta}\right] + e\left(\frac{1}{2b}\right)^2\mathbb{E}\left[\left(a - bc_i + e\mathbb{E}\left[p_{j,2}^c\middle|\,\rho, s_\rho, p_1\right]\right)\frac{\partial}{\partial p_{i,1}}\mathbb{E}\left[p_{j,2}^c\middle|\,\rho, s_\rho, p_1\right]\middle|\,s_\rho, s_{i,\theta}\right].$$

In a linear equilibrium, the first period price is $p_{i,1}^c = p_{0,c} + p_{\theta,c}E[\theta_i|s_{i,\theta}] + p_{\rho,c}E[\rho|s_\rho]$.

[12] For a more through discussion of the simplified model, see **?**.

Because s_ρ and $p_{i,1}$ are publicly observable, then in equilibrium, the value of $s_{i,\theta}$ can be inferred by competing firms. Therefore the expectation of each firm's cost in the second period given publicly available information is $\mathbb{E}[c_i|\rho, s_\rho, p_{i,1}] = \rho + \mathbb{E}[\theta_i|s_{i,\theta}]$, where $s_{i,\theta}$ can be determined from the first period price. Moreover, an increase in the first period price will increase this expectation by the inverse of the equilibrium coefficient $p_{\theta,c}$. The effect of firm i's first period price on firm j's second period price takes into account this informational parameter as well as effects on demand,

$$\frac{\partial}{\partial p_{i,1}}\mathbb{E}\left[p_{j,2}^c\middle|\,\rho, p_{i,1}\right] = \frac{be}{4b^2-e^2}\kappa^c, \text{ where } \kappa^c \equiv \frac{\partial}{\partial p_{i,1}}\mathbb{E}\left[c_i|\,\rho, s_\rho, p_{i,1}\right] = \frac{1}{p_{\theta,c}}$$

In the unique linear equilibrium, $p_{\theta,c}$ is strictly less than p_θ. Therefore firms use less idiosyncratic information in their first period price choice once they have shared common cost information.

Proposition 4. *In the unique equilibrium in linear pricing strategies the coefficient on idiosyncratic information is less than the corresponding coefficient in the equilibrium without information sharing:*

$$p_{\theta,c} = \frac{1-\beta}{2} \leq p_\theta.$$

This inequality is strict when $e \neq 0$. Moreover, the coefficient on common cost information is larger, $p_{\rho,c} \geq p_\rho$, when the firms are selling substitutes ($e \geq 0$).

We can similarly compare the informativeness of first period price about the underlying cost before and after the firms share information on their common cost component. Because $p_{\theta,c} \leq p_\theta$,

$$\beta\kappa = \frac{1-2p_\theta}{p_\theta} \leq \frac{1-2p_{\theta,c}}{p_{\theta,c}} = \frac{\beta}{p_{\theta,c}} = \beta\kappa^c \quad \Longleftrightarrow \quad \kappa \leq \kappa^c.$$

Following from Proposition 4, this inequality is strict when $e \neq 0$.

Once firms have shared common cost information, firms' second period prices are more responsive to the price choices in the first period. In this setting, it is easier to soften future competition and therefore firms have a greater incentive to choose a higher first period price. The increase in expected price imposes a first order negative effect on consumer welfare in the market.

Proposition 5. *Expected first period prices are higher when firms share signals about common cost information, $\mathbb{E}[p_{i,1}] \leq \mathbb{E}[p_{i,1}^c]$. Moreover, expected second period prices are the same regardless of firms sharing information or not.*

From the first order conditions in each case, the only differences in determining the optimal price is the expected price by the competing firm in the second period and the rate at which a first period price increase by a firm affects this second period price by the competitor. The rate of increase is higher when firms share common cost information. The expectation of second period prices does not depend on information sharing. The best response function of each firm is linear in the beliefs about the competing firms costs, and on average, the beliefs must be correct in equilibrium.

4 Information acquisition

In the previous sections we took the precision of the cost signals in the first period of competition as exogenous. We now endogenize the level of precision by allowing firms to pay for more accurate estimates of their individual cost components. We model this as paying a reputable outside research consultant with larger payments leading to more precise results.[13]

Play proceeds as follows: in stage zero, firms pay consultants and consultants provide the firms with (separate) signals of their idiosyncratic and common cost components. The payments and associated precision of information are chosen simultaneously and are not publicly observed. Stages one and two proceed as in Section 3 with the caveat that the precision of signals received is not exogenous as it depends on the level of payment in period zero. Therefore, in period zero firms will choose payments with corresponding levels of precision, $\tau_{i,\theta}, \tau_{i,\rho}$ to maximize

$$\mathbb{E}\left[\mathbb{E}\left[\pi_{i,1}\right| s_i]\right] + \mathbb{E}\left[\mathbb{E}\left[\pi_{i,2}\right| s_i]\right] - c\left(\tau_{i,\theta}, \tau_{i,\rho}\right),$$

were $c\left(\tau_{i,\theta}, \tau_{i,\rho}\right)$ is assumed to be symmetric, separable and convex in each component.

We again focus on symmetric equilibrium. In equilibrium, the level of precision of each firm will be known prior to the two stages of competition and we can apply the analysis of the linear pricing equilibrium obtained in Section 3. In equilibrium the marginal cost of an increase in information precision must equal the benefit that the firms expects to gain in the competition stages from having more precise information. The fact that the acquisition of information is privately observed and application of the envelope theorem allows us to constrain attention to terms of the firms expected profit which vary directly precision choices and ignoring its effect on subsequent choice variables. Additionally, the separability of the firm's profit function allows us to independently analyze the effects of precision on first and second stage profits.

[13]We use this interpretation in part for consistency with Section 4.2, where the single consultant can be thought of as a trade association.

4.1 Expected profits

Expected first stage profits are affected directly by the precision of cost information as it allows firms to make better pricing decisions. However, marginal deviations from equilibrium will not affect the linear pricing coefficients of either firm. By the envelope theorem, marginal changes in precision will not change the optimal pricing strategy of that in the competition stage. The other firm cannot change their pricing strategy based on an unobserved deviation. Therefore, first stage profits can be represented compactly in terms which vary directly with the choice of precision $\tau_{i,x}$ and composite remainder terms which vary only indirectly through $\tau_{i,x}$.

Lemma 6. *There is a function $C_i : \mathbb{R}^6 \to \mathbb{R}$ varying with first-stage price coefficients but otherwise independent of τ_i such that expected first stage profits are given by*

$$\mathbb{E}\left[\pi_{i,1}\right] = \left(\frac{(1-p_{i\theta})\,p_{i\theta}\tau_{i,\theta}}{(\tau_{i,\theta}+\tau_\theta)\,\tau_\theta}\right)b + \left(\frac{(1-p_{i\rho})\,p_{i\rho}\tau_{i,\rho}}{(\tau_{i,\rho}+\tau_\rho)\,\tau_\rho}\right)b - \left(\frac{(1-p_{i\rho})\,p_{j\rho}\tau_{i,\rho}\tau_{j,\rho}}{(\tau_{i,\rho}+\tau_\rho)\,(\tau_{j,\rho}+\tau_\rho)\,\tau_\rho}\right)e + C_i.$$ [14]

First stage profits respond to precision of the two components of marginal cost in a similar way with respect to own demand (the terms postmultiplied by b), but the response differs with respect to cross-firm demand (the term postmultiplied by e). To a first approximation, the precision of the informational signals affects the opponent's payoffs only through information on the common cost component; increasing this precision will increase the correlation in first-period prices, and decreasing this precision will reduce the correlation in first-period prices.

Lemma 7. *The marginal changes in first-stage profits with respect to the precision $\tau_{i,\theta}$ and $\tau_{i,\rho}$ are*

$$\frac{\partial}{\partial\tau_{i,\theta}}\mathbb{E}\left[\pi_{i,1}\right] = \left(\frac{(1-p_{i\theta})\,p_{i\theta}}{(\tau_{i,\theta}+\tau_\theta)^2}\right)b; \tag{1}$$

$$\frac{\partial}{\partial\tau_{i,\rho}}\mathbb{E}\left[\pi_{i,1}\right] = \left(\frac{(1-p_{i\rho})\,p_{i\rho}}{(\tau_{i,\rho}+\tau_\rho)^2}\right)(b-\bar{\tau}_{j,\rho}e). \tag{2}$$

Because all involved terms are positive, it is immediate that when goods are complements (so that $e < 0$) first stage profits respond more strongly to precision on the common cost

[14] The arguments to C_i are omitted for compactness; the equation in Lemma 6 should be written with $C_i(p_{i0}, p_{i\theta}, p_{i\rho}, p_{j0}, p_{j\theta}, p_{j\rho})$. As discussed earlier the dropping of these arguments is unimportant, as C_i does not vary directly with τ_i and thus will be ignored when optimizing over the choice of precision.

component than to the idiosyncratic cost component (holding all else fixed); when goods are substitutes (so that $e > 0$) first stage profits respond more strongly to precision on the idiosyncratic cost component than to the common cost component. As previously discussed, when goods are independent (so that $e = 0$) we have $p_{i\theta} = p_{i\rho}$, implying that first-stage profits respond identically to information on either cost component. If the precisions of the underlying variables θ_i and ρ are not identical, first-stage profits will respond more strongly to increases in precision of the signal of the less precise underlying variable.

A marginal increase in precision does not impact the expected profit in the second stage of competition.[15] Because the firm perfectly observes its cost components after the first stage, increased precision about these parameters has no impact on the information the firm has in the second stage. Additionally, as previously mentioned, neither firm will change first stage strategies. Therefore inferences about the opposing firms cost structure given first stage price does not change. Lastly, because the increase in precision is not observed by the other firm, the inference its makes will also be unaffected. Therefore neither firm's pricing strategy in the second stage will be affected.

From Proposition 3, the first stage pricing coefficient, $p_{i,\theta}$ is less than $1/2$ and is decreasing in $\tau_{i,\theta}$. Therefore the marginal benefit of an increase in $\tau_{i,\theta}$, as shown in Lemma 7, is decreasing. Given the assumptions on the cost of acquisition, this leads to the following partial uniqueness result.

Proposition 6. *Conditional on $\tau_{i,\rho}$, all equilibria in the private information acquisition model have a unique and symmetric choice of $\tau_{i,\theta}$.*

Corollary 1. *$\tau_{i,\theta}$ is increasing in $\tau_{i,\rho}$ when $\bar{\tau}_{i,\rho}$ is close to zero, and decreasing in $\tau_{i,\rho}$ when $\bar{\tau}_{i,\rho}$ is close to one.*

Corollary 1 follows from equation (1). Proposition 3 states that p_θ is decreasing in $\tau_{i,\rho}$ when $\bar{\tau}_{i,\rho}$ is large, and increasing in $\tau_{i,\rho}$ when $\bar{\tau}_{i,\rho}$ is small. Then since $p_\theta \leq 1/2$ for all precision levels, the marginal utility gain from a small increase in $\tau_{i,\rho}$ decreases when $\bar{\tau}_{i,\rho}$ is large and increases when $\bar{\tau}_{i,\rho}$ is small. Then improved precision on the common component leads to decreased precision on the idiosyncratic component when common component precision is already high, and leads to increased precision on the idiosyncratic component when common component precision is low.

Intuitively, when common component precision is high it is difficult for the firm to mask its private signal. Further increasing the precision on the common component makes this even more difficult, and reduces the returns of acquiring private information as the opponent

[15]Note that while marginal deviations in investment in precision cannot affect second-period profits, different levels of believed investment will typically generate different second-period profits.

will learn of this information prior to the second period. In the other direction, when the signal on the common component is relatively imprecise, increasing this precision captures some available profits through improved information — absent signaling incentives, reducing variance is good for the firm. Then since the signal is still relatively imprecise, the firm can still confound its opponent by the use of signaling in its pricing strategy and the information on the idiosyncratic term will be more valuable in the second period

4.2 Sharing industry relevant information

We now consider role of a trade association in the information acquisition stage. The association acts as a pooling resource, producing forecasts about the industry's common cost component from the total payments that it receives from the firms. At a high level this is akin to the trade association producing an article in an industry journal. This information acquisition is modeled as a single technology responsible for improving information on the common cost component. Firms can still pay an additional consultant (or expend resources in house) to gather information on their private cost component.

Again to find the optimal precision of signal we find the effect of an increase in information precision on each of the two periods of competition. Because all information acquired about the common cost component is shared, the two stages of competition will follow as in 3.3. In this setting, pricing strategies in either stage do not depend on information precision and changes in precision does not change the information available to firms in the second stage. Therefore an increase in precision of either the common or idiosyncratic cost component will only affect expect profits in the first stage.

Lemma 8. *There is a constant $C_{i,c}$ which is independent of τ_i such that first-stage profits are given by*

$$\mathbb{E}\left[\pi_{i,1}\left(s_p, s_{i,\theta}\right)\middle|\,\tau_{\varepsilon,\rho}, \tau_{i,\theta}\right] = \left(\frac{\left(1-p_\theta\right)p_\theta\tau_{i,\theta}}{\left(\tau_\theta+\tau_{i,\theta}\right)\tau_\theta}\right)b + \left(\frac{\left(1-p_\rho\right)\tau_{i,\rho}}{\left(\tau_\rho+\tau_{i,\rho}\right)\tau_\rho}\right)(b-e) + C_{i,c}.$$

Lemma 9. *The marginal changes in first-stage profits with respect to the precision $\tau_{i,\theta}$ and $\tau_{c,\rho}$ are*

$$\frac{\partial}{\partial\tau_{i,\theta}}\mathbb{E}\left[\pi^c_{i,1}\right] = \left(\frac{\left(1-p_{\theta,c}\right)p_{\theta,c}}{\left(\tau_{i,\theta}+\tau_\theta\right)^2}\right)b,$$
$$\frac{\partial}{\partial\tau_{c,\rho}}\mathbb{E}\left[\pi^c_{i,1}\right] = \left(\frac{\left(1-p_{\rho,c}\right)p_{\rho,c}}{\left(\tau_{c,\rho}+\tau_\rho\right)^2}\right)(b-e)\,.$$

Lemma 10 (Unique equilibrium)**.** *There is a unique equilibrium in the pooled information model.*

Proof. This follows directly from Lemma 9 and Proposition 4, which states that the unique equilibrium choice of $p_{\theta,c}$ and $p_{\rho,c}$ do not depend on $\tau_{i,\theta}$ and $\tau_{c,\rho}$. □

Proposition 7 (Relative value of information)**.** *The marginal benefit of acquiring additional information in the private cost component is lower when information about the common cost component is gathered by the trade association compared to when common cost information is acquired privately by the two firms,*

$$\frac{\partial}{\partial \tau_{i,\theta}} \mathbb{E}\left[\pi_{i,1}\right] > \frac{\partial}{\partial \tau_{i,\theta}} \mathbb{E}\left[\pi_{i,1}^{c}\right].$$

Proof. Comparing marginal benefits in the first period

$$\frac{\partial}{\partial \tau_{i,\theta}} \mathbb{E}\left[\pi_{i,1}\right] = \left(\frac{\left(1-p_{i\theta}\right) p_{i\theta}}{\left(\tau_{i,\theta}+\tau_{\theta}\right)^{2}}\right) b, \text{ and } \frac{\partial}{\partial \tau_{i,\theta}} \mathbb{E}\left[\pi_{i,1}^{c}\right] = \left(\frac{\left(1-p_{\theta,c}\right) p_{\theta,c}}{\left(\tau_{i,\theta}+\tau_{\theta}\right)^{2}}\right) b.$$

From Proposition 4, $p_{i,\theta}^{c} < p_{i,\theta}^{*} < 1/2$ which implies directly that $\frac{\partial}{\partial \tau_{i,\theta}} \mathbb{E}\left[\pi_{i,1}\right] > \frac{\partial}{\partial \tau_{i,\theta}} \mathbb{E}[\pi_{i,1}^{c}]$. □

Since the marginal benefit of an increase in precision on each stage of profits is lower compared to when there is no informational middleman then firms will gather less information about their private cost component. This may lead to worse outcomes for the firms.

For the common cost component, increased correlation of the signal reduces the value of this signal in the first stage of competition. Moreover, the level of precision has no effect on second stage profits, as the true value of ρ is learned by both firms. Without the informational middleman the precision still affects the second stage due to its signal jamming effect. The direction of this effect is less clear, as increase in the precision of this signal, means that the signal will be closer to the true value and the other firm will become more informed about the signal when they learn ρ. However, a completely uninformative signal does not signal jam at all.

5 Conclusion

We generalize a standard dynamic pricing competition model to allow for uncertainty in common cost and private cost parameters. We characterize the unique symmetric linear equilibrium of this model and use this to examine the incentives of firms to acquire and subsequently share information in this setting. We show that the incentive of firms to acquire additional information about private cost parameters is reduced when firms in the industry pool resources to acquire information about common costs. On the other hand, coordination of information acquisition on idiosyncratic cost terms can alleviate inefficiently low information acquisition. Therefore, the welfare effects of coordination on information between firms depends crucially on the type of information that is acquired.

References

George-Marios Angeletos and Alessandro Pavan. Efficient use of information and social value of information. *Econometrica*, 75(4):1103–1142, 2007.

Dirk Bergemann and Stephen Morris. Robust predictions in games with incomplete information. *Econometrica*, 81(4):1251–1308, 2013.

Dirk Bergemann, Alessandro Bonatti, et al. Markets for information: An introduction. *Cowles Foundation for Research in Economics Discussion paper*, 2018.

Alessandro Bonatti, Gonzalo Cisternas, and Juuso Toikka. Dynamic oligopoly with incomplete information. *The Review of Economic Studies*, 84(2):503–546, 2017.

Luca Colombo, Gianluca Femminis, and Alessandro Pavan. Information acquisition and welfare. *The Review of Economic Studies*, 81(4):1438–1483, 2014.

Thomas D Jeitschko, Ting Liu, and Tao Wang. Information acquisition, signaling and learning in duopoly. *International Journal of Industial Organization*, 2018.

Kai-Uwe Kühn. Fighting collusion by regulating communication between firms. *Economic Policy*, 16(32):168–204, 2001.

Kai-Uwe Kühn and Xavier Vives. Information exchanges among firms and their impact on competition. *manuscript, Institut d'Analisi Economica, Barcelona*, 1994.

George J Mailath. Simultaneous signaling in an oligopoly model. *The Quarterly Journal of Economics*, 104(2):417–427, 1989.

Loretta J Mester. Perpetual signalling with imperfectly correlated costs. *The Rand Journal of Economics*, pages 548–563, 1992.

Leonard J Mirman, Larry Samuelson, and Amparo Urbano. Duopoly signal jamming. *Economic Theory*, 3(1):129–149, 1993.

Leonard J Mirman, Larry Samuelson, and Edward E Schlee. Strategic information manipulation in duopolies. *Journal of Economic Theory*, 62(2):363–384, 1994.

Stephen Morris and Hyun Song Shin. Social value of public information. *American Economic Review*, 92(5):1521–1534, 2002.

David P Myatt and Chris Wallace. Endogenous information acquisition in coordination games. *The Review of Economic Studies*, 79(1):340–374, 2011.

David P Myatt and Chris Wallace. Cournot competition and the social value of information. *Journal of Economic Theory*, 158:466–506, 2015a.

David P Myatt and Chris Wallace. Information use and acquisition in price-setting oligopolies. *The Economic Journal*, 2015b.

Michael Raith. A general model of information sharing in oligopoly. *Journal of economic theory*, 71(1):260–288, 1996.

Xavier Vives. *Oligopoly pricing: old ideas and new tools.* MIT press, 2001.

A Proofs

Proof of Lemma 2

From Lemma 1,

$$p_{i,2}^{\star} = \frac{1}{2b}\left(a + bc_i + e\mathbb{E}\left[p_{j,2}^{\star}\middle|\, \rho, p_{j,1}, p_{i,1}\right]\right).$$

Therefore the expected price of firm j from the perspective of firm i given the public information in the second period is

$$\begin{aligned}\mathbb{E}[p_{j,2}^{\star}|\rho, p_{i,1}, p_{j,1}] &= \frac{1}{2b}\left(a + \left(\mathbb{E}\left[c_j\middle|\, \rho, p_{j,1}, p_{i,1}\right]\right)b + e\mathbb{E}[p_{i,2}^{\star}|\rho, p_1]\right)\\ &= \frac{1}{2b}\left(a + b\mathbb{E}\left[c_j\middle|\, \rho, p_{j,1}, p_{i,1}\right]\right) + \frac{e}{4b^2}\left(a + b\mathbb{E}\left[c_i\middle|\, \rho, p_{j,1}, p_{i,1}\right] + e\mathbb{E}[p_{j,2}^{\star}|\rho, p_1]\right).\end{aligned}$$

Note that firm j's beliefs on firm i's price can only be conditioned on public observables (ρ and first-period prices p_1) and firm i learns nothing [additional] by deviating in the first period, it must be that firm i's expectation of firm j's expectation of firm i's price is equal to the expectation of firms i's price conditioning only on public observables and equilibrium behavior. This leaves

$$\mathbb{E}[p_{j,2}^{\star}|\rho, p_{i,1}, p_{j,1}] = \frac{2b}{4b^2 - e^2}\left[\left(a + b\mathbb{E}\left[c_j\middle|\, \rho, p_{j,1}\right]\right) + \frac{e}{2b}\left(a + b\mathbb{E}\left[c_i\middle|\, \rho, p_{i,1}\right]\right)\right].$$

Proof of Lemma 4

Define $\bar{\tau}_{s,x} = \frac{1/\sigma_{s,x}^2}{1/\sigma_x^2 + 1/\sigma_{s,x}^2}$ and for $x \in \{\theta_i, \theta_j, \rho\}$. Then, in a linear (first stage) equilibrium,

$$\begin{pmatrix} c_i \\ \rho \\ p_{i,1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \bar{\tau}_{s,\theta} p_\theta & \bar{\tau}_{s,\rho} p_\rho \end{pmatrix} \begin{pmatrix} \rho \\ c_i \\ s_{i,\theta} \\ s_{i,\rho} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ p_0 + (1 - \bar{\tau}_{s,\theta})\, \mu_\theta p_\theta + (1 - \bar{\tau}_{s,\rho})\, \mu_\rho p_\rho \end{pmatrix}$$

$$\sim N \left(\begin{pmatrix} \mu_\rho + \mu_\theta \\ \mu_\rho \\ p_0 + p_\theta \mu_\theta + p_\rho \mu_\rho \end{pmatrix}, \begin{pmatrix} \sigma_\rho^2 + \sigma_\theta^2 & \sigma_\rho^2 & \bar{\tau}_{s,\theta} \sigma_\theta^2 p_\theta + \bar{\tau}_{s,\rho} \sigma_\rho^2 p_\rho \\ \sigma_\rho^2 & \sigma_\rho^2 & \bar{\tau}_{s,\rho} \sigma_\rho^2 p_\rho \\ \bar{\tau}_{s,\theta} \sigma_\theta^2 p_\theta + \bar{\tau}_{s,\rho} \sigma_\rho^2 p_\rho & \bar{\tau}_{s,\rho} \sigma_\rho^2 p_\rho & \bar{\tau}_{s,\theta}^2 \left(\sigma_{s,\theta}^2 + \sigma_\theta^2\right) p_\theta^2 + \bar{\tau}_{s,\rho}^2 \left(\sigma_{s,\rho}^2 + \sigma_\rho^2\right) p_\rho^2 \end{pmatrix} \right).$$

It follows that the conditional expectation of c_j given ρ and $p_{j,1}$ is

$$\mathbb{E}\left[c_i \middle| \rho, p_{i,1} \right] = (\mu_\rho + \mu_\theta) + \Sigma_{12} \Sigma_{22}^{-1} \left(\begin{pmatrix} \rho \\ p_{i,1} \end{pmatrix} - \begin{pmatrix} \mu_\rho \\ p_0 + p_\theta \mu_\theta + p_\rho \mu_\rho \end{pmatrix} \right),$$

$$\Sigma_{12} = \begin{pmatrix} \sigma_\rho^2 & \bar{\tau}_{s,\theta} \sigma_\theta^2 p_\theta + \bar{\tau}_{s,\rho} \sigma_\rho^2 p_\rho \end{pmatrix},$$

$$\Sigma_{22} = \begin{pmatrix} \sigma_\rho^2 & \bar{\tau}_{s,\rho} \sigma_\rho^2 p_\rho \\ \bar{\tau}_{s,\rho} \sigma_\rho^2 p_\rho & \bar{\tau}_{s,\theta}^2 \left(\sigma_{s,\theta}^2 + \sigma_\theta^2\right) p_\theta^2 + \bar{\tau}_{s,\rho}^2 \left(\sigma_{s,\rho}^2 + \sigma_\rho^2\right) p_\rho^2 \end{pmatrix}.$$

Letting $\Sigma_{12}\Sigma_{22}^{-1} = (m_1 \;\; m_2)$, it follows that $\kappa_i \equiv \partial \mathbb{E}[\mathbf{c}_i | \rho, p_{i,1}] / \partial p_{i,1} = m_2$; in particular, we only need to care about the right-hand column of Σ_{22}^{-1},

$$\Sigma_{22}^{-1} = \frac{1}{|\Sigma_{22}|} \begin{pmatrix} \cdot & -\bar{\tau}_{s,\rho} \sigma_\rho^2 p_\rho \\ \cdot & \sigma_\rho^2 \end{pmatrix}.$$

It follows that

$$\kappa_i = \frac{\bar{\tau}_{s,\theta} \sigma_\theta^2 \sigma_\rho^2 p_\theta}{|\Sigma_{22}|} = \frac{\bar{\tau}_{s,\theta} \sigma_\theta^2 \sigma_\rho^2 p_\theta}{\bar{\tau}_{s,\theta}^2 \sigma_{s,\theta}^2 \sigma_\rho^2 p_\theta^2 + \bar{\tau}_{s,\rho}^2 \sigma_\rho^2 (\sigma_{s,\rho}^2 + \sigma_\rho^2) p_\rho^2 - \bar{\tau}_{s,\rho}^2 \sigma_\rho^4 p_\rho^2} = \frac{\sigma_\theta^2 \bar{\tau}_{s,\theta} p_\theta}{\sigma_{\varepsilon,\rho}^2 \bar{\tau}_{s,\rho}^2 p_\rho^2 + \left(\sigma_\theta^2 + \sigma_{\varepsilon,\theta}^2\right) \bar{\tau}_{s,\theta}^2 p_\theta^2}.$$

Then

$$\frac{\partial}{\partial p} \mathbb{E}[p_{j,2}^\star | \rho, p_1] = \frac{be}{4b^2 - e^2} \kappa = \frac{be}{4b^2 - e^2} \left(\frac{\sigma_\theta^2 \bar{\tau}_{s,\theta} p_\theta}{\sigma_{s,\rho}^2 \bar{\tau}_{s,\rho}^2 p_\rho^2 + \left(\sigma_\theta^2 + \sigma_{s,\theta}^2\right) \bar{\tau}_{s,\theta}^2 p_\theta^2} \right).$$

Proof of Proposition 1

The first order condition is given by:

$$p_{i,1}^{\star} = \frac{1}{2b}\mathbb{E}\left[bc_i + a + e\hat{p}_{j,1} + \frac{e}{2b}\left(a - bc_i + e\mathbb{E}\left[\left.p_{j,2}^{\star}\right|\rho, p_{i,1}^{\star}, \hat{p}_{j,1}\right]\right)\frac{\partial}{\partial p_{i,1}}\mathbb{E}\left[\left.p_{j,2}^{\star}\right|\rho, p_{i,1}^{\star}, \hat{p}_{j,1}\right]\right| s_{i,\rho}, s_{i,\theta}\right],$$

where

$$\frac{\partial}{\partial p_{i,1}}\mathbb{E}\left[\left.p_{j,2}^{\star}\right|\rho, p_{i,1}^{\star}, \hat{p}_{j,1}\right] = \frac{be}{4b^2 - e^2}\kappa_i$$

and

$$\mathbb{E}\left[\mathbb{E}\left[\left.p_{j,2}^{\star}\right|\rho, p_1\right]\right| s_{i,\rho}, s_{i,\theta}\right] = \frac{2b}{4b^2 - e^2}\left[\left(a + b\mathbb{E}\left[\mathbb{E}\left[c_j|\,\rho, p_{j,1}\right]|\, s_{i,\rho}, s_{i,\theta}\right]\right) + \frac{e}{2b}\left(a + b\mathbb{E}\left[\mathbb{E}\left[c_i|\,\rho, p_{i,1}\right]|\, s_{i,\rho}, s_{i,\theta}\right]\right)\right].$$

In a linear equilibrium, the random variables $(c_k, \rho, p_{k,1})$ are jointly normal and the conditional expectation of cost in the second period is

$$\begin{aligned}\mathbb{E}\left[c_k|\,\rho, p_{k,1}\right] &= (\mu_\rho + \mu_\theta) + \Sigma_{12}\Sigma_{22}^{-1}\left(\begin{pmatrix}\rho \\ p_{k,1}\end{pmatrix} - \begin{pmatrix}\mu_\rho \\ p_0 + p_\theta\mu_\theta + p_\rho\mu_\rho\end{pmatrix}\right) \\ &= (\mu_\rho + \mu_\theta) + \left(1 - \kappa_k\bar{\tau}_{s,\rho,k}p_\rho\right)(\rho - \mu_\rho) + \left(p_{k,1} - (p_0 + p_\rho\mu_\rho + p_\theta\mu_\theta)\right)\kappa_k.\end{aligned}$$

Therefore the expectation of this conditional expectation given the signals available in the first period are

$$\mathbb{E}\left[\mathbb{E}\left[c_j|\,\rho, p_{j,1}\right]|\, s_{i,\rho}, s_{i,\theta}\right] = \mathbb{E}\left[\rho|\, s_{i,\rho}\right] + \mu_\theta \text{ and}$$
$$\mathbb{E}\left[\mathbb{E}\left[c_i|\,\rho, p_{i,1}\right]|\, s_{i,\rho}, s_{i,\theta}\right] = \mathbb{E}\left[\rho|\, s_{i,\rho}\right] + \mu_\theta + \kappa_i p_\rho(1 - \bar{\tau}_{s,\rho,i})\left(\mathbb{E}\left[\rho|\, s_{i,\rho}\right] - \mu_\rho\right) + \kappa_i p_\theta\left(\mathbb{E}\left[\theta_i|\, s_{i,\theta}\right] - \mu_\theta\right),$$

and

$$\begin{aligned}\mathbb{E}\left[\mathbb{E}\left[\left.p_{j,2}^{\star}\right|\rho, p_1\right]\right| s_{i,\rho}, s_{i,\theta}\right] &= \frac{2b}{4b^2 - e^2}\left(a + b\left(\mu_\theta + \mathbb{E}\left[\rho|\, s_{i,\rho}\right]\right)\right) \\ &+ \frac{e}{4b^2 - e^2}\left(a + b\left(\mu_\theta + \kappa_i p_\theta\left(\mathbb{E}\left[\theta_i|\, s_{i,\theta}\right] - \mu_\theta\right) + \mathbb{E}\left[\rho|\, s_{i,\rho}\right] + \kappa_i p_\rho(1 - \bar{\tau}_{s,\rho,i})\left(\mathbb{E}\left[\rho|\, s_{i,\rho}\right] - \mu_\rho\right)\right)\right).\end{aligned}$$

The first order condition becomes

$$\begin{aligned}2bp_{i,1}^{\star} =& \mathbb{E}\left[bc_i + a + ep_{j,1} + \frac{e}{2b}\left(a - bc_i + e\mathbb{E}\left[\left.p_{j,2}^{\star}\right|\rho, p_{i,1}^{\star}, \hat{p}_{j,1}\right]\right)\frac{\partial}{\partial p_{i,1}}\mathbb{E}\left[\left.p_{j,2}^{\star}\right|\rho, p_{i,1}^{\star}, \hat{p}_{j,1}\right]\right| s_{i,\rho}, s_{i,\theta}\right] \\ =& a + e\mathbb{E}\left[p_{j,1}|\, s_{i,\rho}\right] + b\mathbb{E}\left[\mathbf{c}_i|\, s_{i,\rho}, s_{i,\theta}\right] + b\beta\kappa_i\Bigg(\frac{a}{2b - e} + \frac{\left(\mu_\theta + \mathbb{E}\left[\rho|\, s_{i,\rho}\right]\right)be}{4b^2 - e^2} \\ & - \frac{\mathbb{E}\left[\mathbf{c}_i|\, s_{i,\rho}, s_{i,\theta}\right]}{2} + \frac{\beta}{2}\left(\mu_\theta + \kappa_i p_\theta\left(\mathbb{E}\left[\theta|\, s_{i,\theta}\right] - \mu_\theta\right) + \mathbb{E}\left[\rho|\, s_{i,\rho}\right] + \kappa_i p_\rho\left(1 - \bar{\tau}_{s,\rho,i}\right)\left(\mathbb{E}\left[\rho|\, s_{i,\rho}\right] - \mu_\rho\right)\right)\Bigg)\end{aligned}$$

Lastly, note that

$$\begin{aligned}\mathbb{E}\left[p_{j,1}\middle| s_{i,\rho}\right] &= p_0 + \mu_\theta p_\theta + \mathbb{E}\left[\mathbb{E}\left[\rho\middle| s_{j,\rho}\right]\middle| s_{i,\rho}\right] p_{\rho,j} \\ &= p_0 + \mu_\theta p_\theta + \mathbb{E}\left[\bar{\tau}_{s,\rho,j} s_{j,\rho} + \left(1-\bar{\tau}_{s,\rho,j}\right)\mu_\rho\middle| s_{i,\rho}\right] p_{\rho,j} \\ &= p_0 + \mu_\theta p_\theta + \bar{\tau}_{s,\rho,j}\mathbb{E}\left[\rho\middle| s_{i,\rho}\right] p_{\rho,j} + \left(1-\bar{\tau}_{s,\rho,j}\right)\mu_\rho p_{\rho,j}.\end{aligned}$$

Matching coefficients in the first order condition in Lemma 3, price coefficients satisfy the following equalities in any linear equilibrium,

$$\begin{aligned}2bp_0 &= a + \left(p_{0,j} + \mu_\theta p_{\theta,j} + \left(1-\bar{\tau}_{s,\rho,j}\right)\mu_\rho p_{\rho,j}\right) e + \kappa_i b\beta\left(\frac{a}{2b-e} + \frac{\mu_\theta be}{4b^2-e^2}\right) \\ &\quad + \frac{\kappa_i b\beta^2}{2}\left(\left(1-\kappa_i p_\theta\right)\mu_\theta - \left(1-\bar{\tau}_{s,\rho,i}\right)\kappa_i\mu_\rho p_\rho\right), \\ 2bp_\theta &= b + \kappa b\beta\left(-\frac{1}{2} + \frac{1}{2}\beta\kappa p_\theta\right), \qquad (3) \\ 2bp_\rho &= e\bar{\tau}_{s,\rho,j}p_{\rho,j} + b + \kappa b\beta\left(\frac{be}{4b^2-e^2} - \frac{1}{2} + \frac{\beta}{2}\left(1 + \left(1-\bar{\tau}_{s,\rho,i}\right)\kappa p_\rho\right)\right), \qquad (4) \\ \kappa &= \frac{\sigma_\theta^2\bar{\tau}_{s,\theta}p_\theta}{\sigma_{s,\rho}^2\bar{\tau}_{s,\rho}^2p_\rho^2 + \left(\sigma_\theta^2+\sigma_{s,\theta}^2\right)\bar{\tau}_{s,\theta}^2p_\theta^2}. \qquad (5)\end{aligned}$$

Equations (3) and (4) may be equivalently represented as quadratics in $\beta\kappa$, and solving for equilibrium amounts to equating the roots of two quadratic equations. The equations can be reduced to a system of two equations by explicitly solving the p_θ equation.

$$\begin{aligned}2bp_\theta &= b + \frac{\kappa be^2}{4b^2-e^2}\left(-\frac{1}{2} + \frac{1}{2}\left(\frac{e^2}{4b^2-e^2}\right)\kappa p_\theta\right). \\ \Rightarrow 0 &= p_\theta(\beta\kappa)^2 - \beta\kappa + 2(1-2p_\theta).\end{aligned}$$

From equation (3),

$$\beta\kappa = \frac{1}{2p_\theta}\left(1 \pm \sqrt{1-8\left(1-2p_\theta\right)p_\theta}\right) = \frac{1}{2p_\theta}\left(1 \pm \sqrt{\left(1-4p_\theta\right)^2}\right) = \frac{1\pm\left(1-4p_\theta\right)}{2p_\theta}.$$

There are two solutions to this quadratic equation: $\beta\kappa = 2$ and $\beta\kappa = (1-2p_\theta)/p_\theta$. The second order condition is given by

$$-2b + \frac{be^4}{2(4b^2-e^2)^2}\kappa^2 < 0$$

This simplifies to $-2b + b(\beta\kappa)^2/2 < 0$. When $\beta\kappa = 2$ then the left hand side equals zero, so the second order condition is not satisfied. When $\beta\kappa = (1-2p_\theta)/p_\theta$, then $p_\theta = \frac{1}{2+\beta\kappa}$, and the second order condition becomes $p_\theta > 1/6$ which is satisfied in equilibrium.

Solving Equation (4) for p_ρ in a symmetric equilibrium[16] yeilds

$$p_\rho = \frac{b - b\beta\kappa\left(\frac{b-e}{2b+e}\right)}{2b - e\bar{\tau}_{s,\rho} - \frac{1}{2}b\beta^2\kappa^2(1-\bar{\tau}_{s,\rho})}$$

Now, focusing attention on κ, solving equations (3), (4), and (5), and therefore finding an equilibrium reduces to solving the following single-variable equation,

$$\underbrace{(2+\beta\kappa)^2\left(1-\beta\kappa\left(\frac{b-e}{2b-e}\right)\right)^2\sigma_{s,\rho}^2\bar{\tau}_{s,\rho}^2\kappa}_{\text{LHS}(\kappa)} = \underbrace{\left(\left(2-\frac{e}{b}\bar{\tau}_{s,\rho}\right)-\frac{1}{2}\left(1-\bar{\tau}_{s,\rho}\right)\beta^2\kappa^2\right)^2\left(2-\left(1-\beta\right)\kappa\right)\sigma_\theta^2\bar{\tau}_{s,\theta}}_{\text{RHS}(\kappa)}. \tag{6}$$

To see that an equilibrium exists, let $\bar{\kappa} = 2/(1-\beta)$ be the maximum feasible value of κ and note that $\text{LHS}(0) = 0$ and $\text{RHS}(0) > 0$, and $\text{LHS}(\bar{\kappa}) > 0$ and $\text{RHS}(\bar{\kappa}) = 0$. Since both sides of the equation are continuous in κ, there exists a κ that solves the equation, and this κ will determine the linear pricing parameters p_0, p_θ, and p_ρ.

Showing uniqueness is more involved. We first show that RHS is decreasing. We then show that either LHS is increasing, or it is increasing and then decreasing. Where LHS is decreasing, it is concave and RHS is convex. Since $\text{LHS}(\bar{\kappa}) > \text{RHS}(\bar{\kappa})$ when $\bar{\tau}_{s,\rho} > 0$, there is a unique crossing point whenever $\bar{\tau}_{s,\rho} > 0$. When $\bar{\tau}_{s,\rho} = 0$, LHS is identically 0 for all κ, so there is a unique crossing point at $\kappa = \bar{\kappa}$. In either case, there is a unique feasible value of κ that solves the equilibrium sufficient condition.

First, RHS is decreasing. This can be observed directly, and calculus is not necessary. By inequalities (33) and (37), $1-\beta \in [0, 2/3]$ and $\kappa \in [0, 3]$, so $2-(1-\beta)\kappa \geq 0$ is decreasing. By inequality (39), $\beta\kappa \in [0, 1]$, so $(2-\bar{\tau}_{s,\rho}e/b) - (1-\bar{\tau}_{s,\rho})\beta^2\kappa^2/2 > 0$ is decreasing. Then RHS is the product of two decreasing and positive functions, and is itself decreasing.

Second, LHS is either increasing, or is increasing and then decreasing. The first derivative

[16] Here we impose both $p_{\rho,j} = p_\rho$ and $\bar{\tau}_{s,\rho,i} = \bar{\tau}_{s,\rho,j}$.

of LHS is

$$
\begin{aligned}
\frac{\partial \text{LHS}}{\partial \kappa} &= 2\beta\left(2+\beta\kappa\right)\left(1-\beta\kappa\left(\frac{b-e}{2b-e}\right)\right)^2 \sigma_{s,\rho}^2 \bar{\tau}_{s,\rho}^2 \kappa \\
&\quad - 2\beta\left(\frac{b-e}{2b-e}\right)\left(2+\beta\kappa\right)^2\left(1-\beta\kappa\left(\frac{b-e}{2b-e}\right)\right)\sigma_{s,\rho}^2 \bar{\tau}_{s,\rho}^2 \kappa \\
&\quad + \left(2+\beta\kappa\right)^2\left(1-\beta\kappa\left(\frac{b-e}{2b-e}\right)\right)^2 \sigma_{s,\rho}^2 \bar{\tau}_{s,\rho}^2 .
\end{aligned}
$$

Factoring out common positive terms gives

$$
\begin{aligned}
\frac{\partial \widetilde{\text{LHS}}}{\partial \kappa} &= 2\beta\left(1-\beta\kappa\left(\frac{b-e}{2b-e}\right)\right)\kappa - 2\beta\left(\frac{b-e}{2b-e}\right)\left(2+\beta\kappa\right)\kappa + \left(2+\beta\kappa\right)\left(1-\beta\kappa\left(\frac{b-e}{2b-e}\right)\right) \\
&= 2 + \left(2\beta - 4\beta\left(\frac{b-e}{2b-e}\right) + \beta - 2\beta\left(\frac{b-e}{2b-e}\right)\right)\kappa \\
&\quad + \left(-2\beta^2\left(\frac{b-e}{2b-e}\right) - 2\beta^2\left(\frac{b-e}{2b-e}\right) - \beta^2\left(\frac{b-e}{2b-e}\right)\right)\kappa^2 \\
&= 2 + \left(3 - 6\left(\frac{b-e}{2b-e}\right)\right)\beta\kappa - 5\left(\frac{b-e}{2b-e}\right)\beta^2\kappa^2 . \qquad (7)
\end{aligned}
$$

Then the sign of $\partial \text{LHS}/\partial\kappa$ is determined by the sign of $\partial\widetilde{\text{LHS}}/\partial\kappa$, which is a negative quadratic in κ. Since $\text{LHS}(0) = 0$ and LHS is positive, it follows that either LHS is increasing, or it is increasing and then decreasing.

When LHS is increasing and then decreasing, its inflection point $\kappa^{\perp}$ will be given by one of the zeros of equation (7).[17] The quadratic equation gives these zeros as

$$
\begin{aligned}
\beta\kappa^{\perp} &= \frac{2b-e}{10\left(b-e\right)}\left(\left(3-\left(\frac{b-e}{2b-e}\right)\right) \pm \sqrt{\left(3-6\left(\frac{b-e}{2b-e}\right)\right)^2 + 40\left(\frac{b-e}{2b-e}\right)}\right) \\
&= \frac{1}{10\left(b-e\right)}\left(\left(3\left(2b-e\right)-6\left(b-e\right)\right) \pm \sqrt{\left(3\left(2b-e\right)-6\left(b-e\right)\right)^2 + 40\left(b-e\right)\left(2b-e\right)}\right) \\
&= \frac{1}{10\left(b-e\right)}\left(3e \pm \sqrt{\left(3e\right)^2 + 40\left(2b^2-3be+e^2\right)}\right).
\end{aligned}
$$

Since $\kappa \geq 0$ and the radicand is weakly larger than $(3e)^2$, only the "plus" solution to the quadratic is valid. Letting $r_{eb} = e/b$ and treating this as a quadratic in $\beta\kappa$, this is

$$
\beta\kappa^{\perp} = \frac{1}{10-10r_{eb}}\left(3r_{eb} + \sqrt{80-120r_{eb}+49r_{eb}^2}\right).
$$

[17] We analyze equation (7) as a quadratic in $\beta\kappa$. Since we will show that $\beta\kappa$ is decreasing in r_{eb} and β is increasing in r_{eb}, it follows that κ is decreasing in r_{eb}.

Taking the derivative of $\kappa^{\perp}$ with respect to r_{eb} gives

$$\frac{\partial \beta\kappa^{\perp}}{\partial r_{eb}} = \left(\frac{1}{10-10r_{eb}}\right)^2 \left(\left(3+\frac{49r_{eb}-60}{\sqrt{80-120r_{eb}+49r_{eb}^2}}\right)(10-10r_{eb}) \right. \\ \left. +10\left(3r_{eb}+\sqrt{80-120r_{eb}+49r_{eb}^2}\right)\right). \tag{8}$$

We want to show that $\kappa^{\perp}$ is minimized when $e=-b$, so we check that $\partial\beta\kappa^{\perp}/\partial r_{eb} > 0$; it is sufficient to check the numerator of equation (8).

$$\begin{aligned} 0 &\gtrless \left(3+\frac{49r_{eb}-60}{\sqrt{80-120r_{eb}+49r_{eb}^2}}\right)(10-10r_{eb}) \\ &\quad + 10\left(3r_{eb}+\sqrt{80-120r_{eb}+49r_{eb}^2}\right) \\ \iff 0 &\gtrless \left(3\sqrt{80-120r_{eb}+49r_{eb}^2}+49r_{eb}-60\right)(1-r_{eb}) \\ &\quad + \left(3r_{eb}\sqrt{80-120r_{eb}+49r_{eb}^2}+80-120r_{eb}+49r_{eb}^2\right) \\ \iff &-3\sqrt{80-120r_{eb}+49r_{eb}^2} \\ &\gtrless (49r_{eb}-60)(1-r_{eb}) + \left(80-120r_{eb}+49r_{eb}^2\right) = 20-11r_{eb}. \end{aligned}$$

Since the left-hand side of the final inequality is always negative and the right-hand side is always positive, it is the case that $\partial\beta\kappa^{\perp}/\partial r_{eb} > 0$. Then $\beta\kappa^{\perp}$ is minimized at $r_{eb}=-1$, or $e=-b$. This gives

$$\beta\kappa^{\perp} = \frac{1}{20}\left(-3+\sqrt{80+120+49}\right) \implies \kappa^{\perp} = \frac{3}{20}\left(-3+\sqrt{249}\right) \approx 1.917 \geq \frac{3}{2}.$$

Then when LHS is decreasing, it is only decreasing for $\kappa > 3/2$.

We now show that where LHS is decreasing, it is concave. Placing the common positive terms back into equation (7) gives

$$\frac{\partial \text{LHS}}{\partial \kappa} \propto \left(2+\left(3-6\left(\frac{b-e}{2b-e}\right)\right)\beta\kappa - 5\left(\frac{b-e}{2b-e}\right)\beta^2\kappa^2\right)(2+\beta\kappa)\left(1-\beta\kappa\left(\frac{b-e}{2b-e}\right)\right). \tag{9}$$

Two points are of note: first, equation (7) implies $[\partial\widetilde{\text{LHS}}/\partial\kappa]|_{e=0} > 0$, so if LHS is decreasing our earlier result showing that $\kappa^{\perp}$ is increasing in r_{eb} implies that $e<0$. Second, the left two terms in relationship 9 reduce to

$$(2+\beta\kappa)\left(1-\beta\kappa\left(\frac{b-e}{2b-e}\right)\right)=2+\left(\frac{e}{2b-e}\right)\beta\kappa-\left(\frac{b-e}{2b-e}\right)\beta^2\kappa^2.$$

This product is a negative quadratic that is maximized at $\kappa = -e/(2\beta(2b-e))$. If this quantity is greater than 3, the above equation is positive and increasing for all relevant κ. Since $\partial\widetilde{\text{LHS}}/\partial\kappa$ is negative and decreasing (increasing in magnitude), this implies that $\partial^2\text{LHS}/\partial\kappa < 0$ and LHS is concave. For $-e/(2\beta(2b-e)) \geq 3$ we require

$$-\frac{e}{2\beta\left(2b-e\right)}\geq 3 \iff -\frac{1}{2}\left(\frac{2b+e}{e}\right)\geq 3 \iff e\leq -\frac{2}{7}b \iff r_{eb}\leq -\frac{2}{7}.$$

By definition, there is a decreasing portion of LHS if and only if $\kappa^{\perp} < 3$. Then we need to show that $\kappa^{\perp} < 3$ implies $r_{eb} \leq -2/7$. Since we have shown that $\kappa^{\perp}$ is increasing in r_{eb}, it is sufficient to show that $r_{eb} = -2/7$ implies $\kappa^{\perp} \geq 3$. To see this, substitute into equation (7) with $r_{eb} = -2/7$,

$$\left.\frac{\partial\widetilde{\text{LHS}}}{\partial\kappa}\right|_{r_{eb}=-\frac{2}{7}}=2+\left(3-6\left(\frac{\frac{9}{7}}{\frac{16}{7}}\right)\right)\beta\kappa-5\left(\frac{\frac{9}{7}}{\frac{16}{7}}\right)\beta^2\kappa^2\propto 32-6\beta\kappa-45\beta^2\kappa^2.$$

Solving this quadratic in $\beta\kappa$ yields

$$\beta\kappa=-\frac{1}{90}\left(6+\sqrt{36+4\left(45\right)\left(32\right)}\right)=-\frac{1}{15}\left(1\pm\sqrt{161}\right).$$

Since $\beta\kappa \geq 0$, only the "minus" solution is negative. This yields

$$\begin{aligned}\kappa=\frac{1}{15}\left(\sqrt{161}-1\right)\left(\frac{1}{\beta}\right)&=\frac{1}{15}\left(\sqrt{161}-1\right)\left(\frac{4-r_{eb}^2}{r_{eb}^2}\right)\\&=\frac{1}{15}\left(\sqrt{161}-1\right)\left(\frac{4-\left(-\frac{2}{7}\right)^2}{\left(-\frac{2}{7}\right)^2}\right)=\frac{16}{5}\left(\sqrt{161}-1\right)\approx 37.403.\end{aligned}$$

Then $r_{eb} = -2/7$ implies $\kappa^{\perp} > 3$, hence $\kappa^{\perp} \leq 3$ implies $r_{eb} < -2/7$. Then LHS is concave when it is decreasing.

We now show that RHS is convex for $\kappa \in [3/2, 3]$, thus when LHS is possibly decreasing RHS is convex. We show this directly by examining the second derivative and showing that it is positive.

$$\begin{aligned}\frac{\partial \text{RHS}}{\partial \kappa} \propto & -2\left(1-\bar{\tau}_{s,\rho}\right)\beta^2\kappa\left(\left(2-\frac{e}{b}\bar{\tau}_{s,\rho}\right)-\frac{1}{2}\left(1-\bar{\tau}_{s,\rho}\right)\beta^2\kappa^2\right)\left(2-\left(1-\beta\right)\kappa\right) \\ & -\left(1-\beta\right)\left(\left(2-\frac{e}{b}\bar{\tau}_{s,\rho}\right)-\frac{1}{2}\left(1-\bar{\tau}_{s,\rho}\right)\beta^2\kappa^2\right)^2;\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \text{RHS}}{\partial \kappa^2} \propto & -2\left(1-\bar{\tau}_{s,\rho}\right)\beta^2\left(\left(2-\frac{e}{b}\bar{\tau}_{s,\rho}\right)-\frac{1}{2}\left(1-\bar{\tau}_{s,\rho}\right)\beta^2\kappa^2\right)\left(2-\left(1-\beta\right)\kappa\right) \\ & +2\left(1-\bar{\tau}_{s,\rho}\right)^2\beta^4\kappa^2\left(2-\left(1-\beta\right)\kappa\right) \\ & +2\left(1-\beta\right)\left(1-\bar{\tau}_{s,\rho}\right)\beta^2\kappa\left(\left(2-\frac{e}{b}\bar{\tau}_{s,\rho}\right)-\frac{1}{2}\left(1-\bar{\tau}_{s,\rho}\right)\beta^2\kappa^2\right) \\ & +2\left(1-\beta\right)\left(1-\bar{\tau}_{s,\rho}\right)\beta^2\kappa\left(\left(2-\frac{e}{b}\bar{\tau}_{s,\rho}\right)-\frac{1}{2}\left(1-\bar{\tau}_{s,\rho}\right)\beta^2\kappa^2\right) \\ \propto & -\left(\left(2-\frac{e}{b}\bar{\tau}_{s,\rho}\right)-\frac{1}{2}\left(1-\bar{\tau}_{s,\rho}\right)\beta^2\kappa^2\right)\left(2-\left(1-\beta\right)\kappa\right) \\ & +\beta^2\kappa^2\left(2-\left(1-\beta\right)\kappa\right)+2\left(1-\beta\right)\kappa\left(\left(2-\frac{e}{b}\bar{\tau}_{s,\rho}\right)-\frac{1}{2}\left(1-\bar{\tau}_{s,\rho}\right)\beta^2\kappa^2\right) \\ = & \left(3\left(1-\beta\right)\kappa-2\right)\left(\left(2-\frac{e}{b}\bar{\tau}_{s,\rho}\right)-\frac{1}{2}\left(1-\bar{\tau}_{s,\rho}\right)\beta^2\kappa^2\right)+\beta^2\kappa^2\left(2-\left(1-\beta\right)\kappa\right).\end{aligned}$$

By definition $1-\beta \geq 2/3$, and by assumption $\kappa > 3/2$. Then $3(1-\beta)\kappa > 3$. It follows that

$$\frac{\partial^2 \text{RHS}}{\partial \kappa^2} \geq \left(\left(\left(2-\frac{e}{b}\bar{\tau}_{s,\rho}\right)-\frac{1}{2}\left(1-\bar{\tau}_{s,\rho}\right)\beta^2\kappa^2\right)+\beta^2\kappa^2\left(2-\left(1-\beta\right)\kappa\right)\right)\sigma_\theta^2\bar{\tau}_{s,\theta} \geq 0.$$

Moreover, this inequality is strict so long as $\sigma_\theta^2\bar{\tau}_{s,\theta} > 0$. Then RHS is convex for $\kappa > 3/2$.

Putting these arguments together gives the following. LHS and RHS intersect on the interval $\kappa \in [0,3]$. RHS is decreasing, and if LHS is increasing they have a unique intersection. If LHS is not increasing, it is increasing for small values of κ and decreasing for large values of κ, and where it is decreasing it is concave. For any possible values on which LHS is decreasing, RHS is convex. Then there is at most one intersection on the region where LHS is decreasing. If LHS is greater than RHS at the point at which it becomes decreasing, the curves do not intersect at any point to the right because $\text{LHS}(\bar{\kappa}) > \text{RHS}(\bar{\kappa}) = 0$. Then the curves intersect at a unique point, and equilibrium κ is unique. This directly implies that equilibrium price strategies are unique.

Proof of Proposition 2

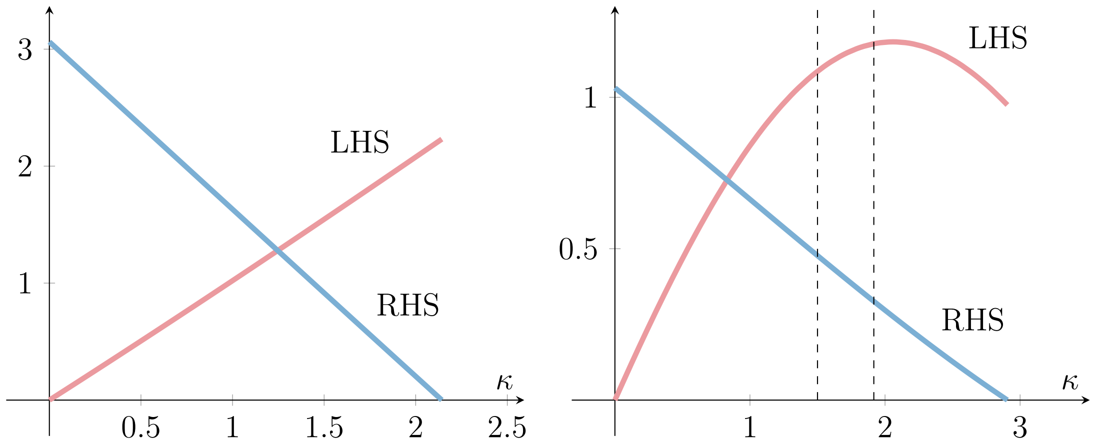

Figure 1: A graphical depiction of the proof of equilibrium existence and uniqueness. The existence of an equilibrium amounts to finding a κ such that LHS(κ) = RHS(κ). Since LHS(0) < RHS(0) and LHS($\bar{\kappa}$) > RHS($\bar{\kappa}$) and both functions are continuous, such a κ is guaranteed to exist. Additionally, RHS is decreasing. We show that either LHS is increasing (left panel) or increasing and then decreasing (right panel). In the former case, it is clear that there is a unique point of intersection and hence a unique equilibrium. In the latter case, we show that LHS is concave where it is decreasing and RHS is convex anywhere LHS is decreasing. Then LHS − RHS is concave, ensuring that equilibrium κ is unique.

From Proposition 1 we know that the values of p_θ and p_ρ in equilibrium are

$$p_\rho = \frac{1 - \beta\kappa\left(\frac{b-e}{2b-e}\right)}{2 - \frac{e}{b}\bar{\tau}_{s,\rho} - \frac{1}{2}\beta^2\kappa^2(1-\bar{\tau}_{s,\rho})} \text{ and } p_\theta = \frac{1}{2+\beta\kappa}.$$

If we multiply the top and bottom of the expression for p_θ by $1 - \beta\kappa\left(\frac{b-e}{2b-e}\right)$, then the numerators are the same and we only need to compare the denominators of each expression. The denominator of p_θ becomes

$$(2+\beta\kappa)\left(1 - \beta\kappa\left(\frac{b-e}{2b-e}\right)\right) = 2 - \left(\frac{b-e}{2b-e}\right)\beta^2\kappa^2 - 2\left(\frac{b-e}{2b-e}\right)\beta\kappa + \beta\kappa$$

When $0 < e \le b$, then $\left(\frac{b-e}{2b-e}\right) \in \left[0, \frac{1}{2}\right)$, and

$$\begin{aligned} 2 - \left(\frac{b-e}{2b-e}\right)\beta^2\kappa^2 - 2\left(\frac{b-e}{2b-e}\right)\beta\kappa + \beta\kappa &\ge 2 - \frac{1}{2}\beta^2\kappa^2(1-\bar{\tau}_{s,\rho}) - \frac{1}{2}\beta^2\kappa^2\bar{\tau}_{s,\rho} \\ &> 2 - \frac{e}{b}\bar{\tau}_{s,\rho} - \frac{1}{2}\beta^2\kappa^2(1-\bar{\tau}_{s,\rho}). \end{aligned}$$

The last inequality follows from the fact that

$$\frac{1}{2}\beta^2\kappa^2 \leq \frac{1}{2}\beta\kappa = \frac{e}{b}\frac{be}{2(4b^2-e^2)}\kappa < \frac{e}{b}.$$

Since the denominator of p_θ is at least as large as that of p_ρ, then $p_\rho < p_\theta$.

When $-b \leq e < 0$, then $\left(\frac{b-e}{2b-e}\right) \in \left[\frac{1}{2}, \frac{2}{3}\right)$, and $\frac{e}{b} \leq \frac{1}{2}\beta^2\kappa^2$. Therefore the above inequalities flip and for these range of parameters, $p_\rho \leq p_\theta$.

Proof of Proposition 3

Effect of $\bar{\tau}_{s,\theta}$

From Proposition 1 we know that there is a unique $\kappa \in \left[0, \frac{2}{1-\beta}\right]$ such that the LHS and the RHS of equation (6) are equal. Moreover, for $\kappa = 0$ the LHS is 0 and the right hand side is positive, and for $\kappa = \frac{2}{1-\beta}$ the RHS is 0 and the LHS is positive. Therefore, for all κ larger than the κ which satisfies the equation, the LHS is larger than the RHS.

As $\bar{\tau}_{s_\theta}$ increases, the LHS of the equation is constant while the RHS increases for all κ. This shift up of the RHS will increase the κ which satisfies the equation. Because of the inverse relationship between κ and p_θ, p_θ will decrease as as $\bar{\tau}_{s,\theta}$ increases.

Effect of $\bar{\tau}_{s,\rho}$

From our definition of $\bar{\tau}_{s,x}$, we have

$$\bar{\tau}_{s,x} = \frac{\sigma_x^2}{\sigma_x^2 + \sigma_{s,x}^2} \quad \implies \quad \sigma_{s,x}^2 = \left(\frac{1-\bar{\tau}_{s,x}}{\bar{\tau}_{s,x}}\right)\sigma_x^2.$$

Then in the fixed point equation,

$$\text{LHS}(\kappa) = (2+\beta\kappa)^2\left(1-\beta\kappa\left(\frac{b-e}{2b-e}\right)\right)^2(1-\bar{\tau}_{s,\rho})\,\bar{\tau}_{s,\rho}\sigma_\rho^2\kappa, \text{ and}$$
$$\frac{\partial\text{LHS}}{\partial\bar{\tau}_{s,\rho}}(\kappa) = (2+\beta\kappa)^2\left(1-\beta\kappa\left(\frac{b-e}{2b-e}\right)\right)^2(1-2\bar{\tau}_{s,\rho})\,\sigma_\rho^2\kappa.$$

The first equality follows directly from the definition of LHS and the above equation for $\sigma_{s,\rho}^2$. In particular, $\sigma_{s,\rho}^2\bar{\tau}_{s,\rho}^2 = (1-\bar{\tau}_{s,\rho})\,\bar{\tau}_{s,\rho}\sigma_\rho^2$. The second equality is then immediate. This implies that $\partial\text{LHS}/\partial\bar{\tau}_{s,\rho}$ is linear and decreasing in $\bar{\tau}_{s,\rho}$, and is positive when $\bar{\tau}_{s,\rho} < 1/2$ and negative when $\bar{\tau}_{s,\rho} > 1/2$.

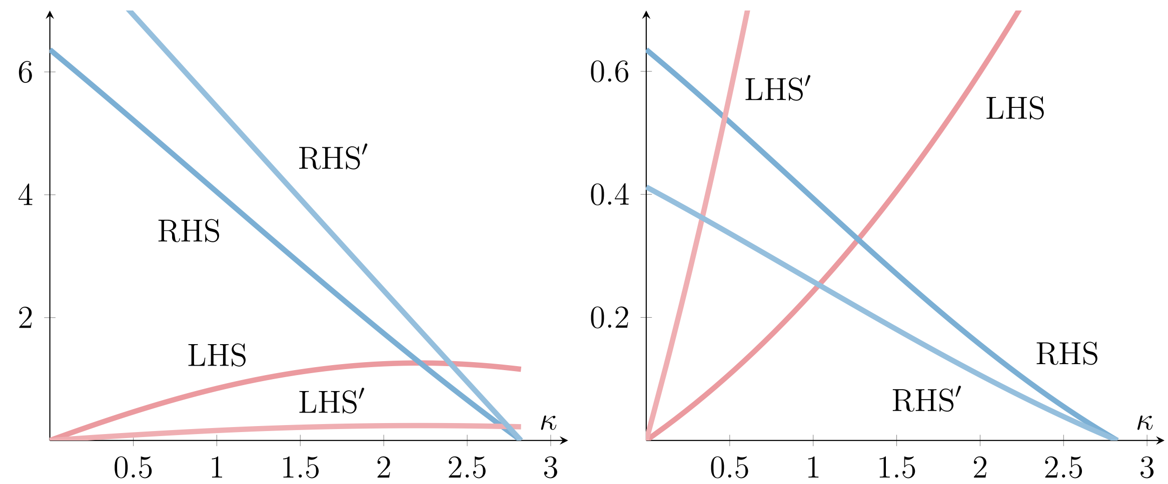

Figure 2: A graphical depiction of the response of κ to $\bar{\tau}_{s,\rho}$. When $\bar{\tau}_{s,\rho} \geq 1/2$ and $e < 0$, an increase in $\bar{\tau}_{s,\rho}$ increases RHS and decreases LHS, pushing κ rightward (left panel). When $\bar{\tau}_{s,\rho} \leq 1/2$ and $e > 0$, an increase in $\bar{\tau}_{s,\rho}$ decreases RHS and increases LHS, pushing κ leftward (right panel).

Additionally in the fixed point equation,

$$\text{RHS}(\kappa) = \left(2b - \frac{1}{2}b\beta^2\kappa^2 + \left(\frac{1}{2}b\beta^2\kappa^2 - e\right)\bar{\tau}_{s,\rho}\right)^2 \left(2 - (1-\beta)\kappa\right)\sigma_\theta^2 \bar{\tau}_{s,\theta}, \text{ and}$$

$$\frac{\partial \text{RHS}}{\partial \bar{\tau}_{s,\rho}}(\kappa) = 2\left(\frac{1}{2}b\beta^2\kappa^2 - e\right)\left(2b - \frac{1}{2}b\beta^2\kappa^2 + \left(\frac{1}{2}b\beta^2\kappa^2 - e\right)\bar{\tau}_{s,\rho}\right)\left(2 - (1-\beta)\kappa\right)\sigma_\theta^2 \bar{\tau}_{s,\theta}.$$

Therefore $\partial\text{RHS}/\partial\bar{\tau}_{s,\rho}$ is linear and increasing in $\bar{\tau}_{s,\rho}$, and is positive when $e < 0$ and negative when $e > 0$. This follows directly from rearrangement of the internal term,

$$\frac{1}{2}b\beta^2\kappa^2 - e = \left(\frac{1}{2}(\beta\kappa)\left(\frac{be}{4b^2 - e^2}\kappa\right) - 1\right)e.$$

If $e < 0$, $be/(4b^2 - e^2) < 0$; if $e > 0$, $\beta\kappa < 1$ and $be\kappa/(4b^2 - e^2) < 1$, so difference in parentheses is negative. In either case, the term is signed as $-e$. Then when $e < 0$ this term is positive and when $e > 0$ this term is negative. Lemma 11 follows directly from the above observations.

Lemma 11. *When $\bar{\tau}_{s,\rho} \geq 1/2$ and $e < 0$, κ is increasing and p_θ is decreasing in $\bar{\tau}_{s,\rho}$. When $\bar{\tau}_{s,\rho} \leq 1/2$ and $e > 0$, κ is decreasing and p_θ is increasing in $\bar{\tau}_{s,\rho}$.*

Lemma 12 (Limiting cases of common information)**.** *When $\bar{\tau}_{s,\rho} \in \{0, 1\}$,*

$$\kappa = \frac{1}{p_\theta} \implies p_\theta = \frac{1-\beta}{2}.$$

Proof. The second equality follows directly from the first. Recall that $\sigma_{s,\rho}^2 \bar{\tau}_{s,\rho}^2 = (1 - \bar{\tau}_{s,\rho}) \bar{\tau}_{s,\rho} \sigma_\rho^2$. Since all endogenous terms are bounded, when $\bar{\tau}_{s,\rho} \in \{0, 1\}$ we have

$$\kappa = \frac{\sigma_\theta^2 \bar{\tau}_{s,\theta} p_\theta}{\sigma_{s,\rho}^2 \bar{\tau}_{s,\rho}^2 p_\rho^2 + \left(\sigma_\theta^2 + \sigma_{s,\theta}^2\right) \bar{\tau}_{s,\theta}^2 p_\theta^2} = \frac{\sigma_{s,\rho}^2 \bar{\tau}_{s,\theta} p_\theta}{\left(\sigma_\theta^2 + \sigma_{s,\theta}^2\right) \bar{\tau}_{s,\theta}^2 p_\theta^2} = \frac{1}{p_\theta}.$$

□

Lemma 13 (Effect of $\bar{\tau}_{s,\rho}$ on p_θ (first hard case))**.** *When $e > 0$, there is a $\tau^\star > 1/2$ such that whenever $\bar{\tau}_{s,\rho} > \tau^\star$, κ is increasing and p_θ is decreasing in $\bar{\tau}_{s,\rho}$, and whenever $\bar{\tau}_{s,\rho} < \tau^\star$, κ is decreasing and p_θ is increasing in $\bar{\tau}_{s,\rho}$.*

Proof. This is a loose proof.

We have already established that if such a $\tau^\star$ exists, it is greater than $1/2$. Since $p_\theta = (1-\beta)/2$ when $\bar{\tau}_{s,\rho} \in \{0, 1\}$ and κ is continuous in the parameters of the problem, there is a value $\hat{\tau}^\star$ such that $\partial\text{LHS}/\partial\bar{\tau}_{s,\rho} = \partial\text{RHS}/\partial\bar{\tau}_{s,\rho}$ at $\hat{\tau}^\star$. Because both of these partial derivatives are linear, a non-marginal increase in $\bar{\tau}_{s,\rho}$ leads to a point where $\partial\text{RHS}/\partial\bar{\tau}_{s,\rho} > \partial\text{LHS}/\partial\bar{\tau}_{s,\rho}$; as discussed earlier, a non-marginal increase in $\bar{\tau}_{s,\rho}$ from this point leads (naïvely) to an increase in κ.

We can additionally show that when $e > 0$ and $\bar{\tau}_{s,\rho} > 1/2$, $\partial\text{RHS}/\partial\bar{\tau}_{s,\rho}$ is increasing and $\partial\text{LHS}/\partial\bar{\tau}_{s,\rho}$ is decreasing in κ. This add-on effect implies that $\partial\text{RHS}/\partial\bar{\tau}_{s,\rho}$ is even larger and $\partial\text{LHS}/\partial\bar{\tau}_{s,\rho}$ is even smaller than we would naïvely expect, pushing κ even higher. Then all effects point in the same direction. Letting $\tau^\star = \hat{\tau}^\star$ completes the proof. □

Lemma 14 (Effect of $\bar{\tau}_{s,\rho}$ on p_θ (first hard case))**.** *When $e < 0$, there is a $\tau^\star \leq 1/2$ such that whenever $\bar{\tau}_{s,\rho} < \tau^\star$, κ is deceasing and p_θ is increasing in $\bar{\tau}_{s,\rho}$, and whenever $\bar{\tau}_{s,\rho} > \tau^\star$, κ is increasing and p_θ is decreasing in $\bar{\tau}_{s,\rho}$.*

Proof. This is a loose proof.

We have already established that if such a $\tau^\star$ exists, it is less than $1/2$. As noted earlier, $\partial\text{LHS}/\partial\bar{\tau}_{s,\rho}$ is positive and decreasing in $\bar{\tau}_{s,\rho}$ and $\partial\text{RHS}/\partial\bar{\tau}_{s,\rho}$ is positive and increasing in $\bar{\tau}_{s,\rho}$. If $\partial\text{LHS}/\partial\bar{\tau}_{s,\rho} > \partial\text{RHS}/\partial\bar{\tau}_{s,\rho}$, then κ is locally decreasing in $\bar{\tau}_{s,\rho}$.

From earlier results, we have

$$\frac{\partial\text{LHS}}{\partial\bar{\tau}_{s,\rho}}(\kappa) = \left(\frac{1 - 2\bar{\tau}_{s,\rho}}{(1 - \bar{\tau}_{s,\rho})\bar{\tau}_{s,\rho}}\right) \text{LHS}(\kappa), \text{ and}$$

$$\frac{\partial\text{RHS}}{\partial\bar{\tau}_{s,\rho}}(\kappa) = \left(\frac{b\beta^2\kappa^2 - 2e}{2b - \frac{1}{2}b\beta^2\kappa^2 + \left(\frac{1}{2}b\beta^2\kappa^2 - e\right)\bar{\tau}_{s,\rho}}\right) \text{RHS}(\kappa).$$

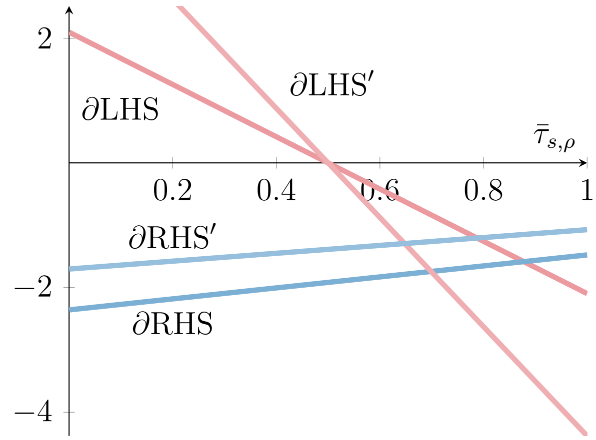

Figure 3: A graphical depiction of the comparative statics of κ with respect to $\bar{\tau}_{s,\rho}$ when $e > 0$ and $\bar{\tau}_{s,\rho} > 1/2$. In this case, both LHS and RHS decrease. We show that for sufficiently high $\bar{\tau}_{s,\rho}$, LHS decreases at a faster rate than RHS. The existence of a unique threshold determining which term decreases faster follows from the observation that from the point at which they decrease equally (the $\bar{\tau}_{s,\rho}$ at which $\partial\kappa/\partial\bar{\tau}_{s,\rho} = 0$), a slight increase in $\bar{\tau}_{s,\rho}$ moves into a region where LHS is decreasing faster than RHS. While the first-order effect is zero, the second-order effect implies that κ is increasing. We show that in this case, $\partial\text{LHS}/\partial\bar{\tau}_{s,\rho}$ is decreasing in κ while $\partial\text{RHS}/\partial\bar{\tau}_{s,\rho}$ is increasing in κ, so the additional second-order effect (due to the derivatives shifting) points in the same direction.

In any equilibrium, $\text{LHS}(\kappa) = \text{RHS}(\kappa)$, so checking $\partial\text{LHS}/\partial\bar{\tau}_{s,\rho} > \partial\text{RHS}/\partial\bar{\tau}_{s,\rho}$ is equivalent to checking

$$\begin{aligned}
& & \frac{1 - 2\bar{\tau}_{s,\rho}}{(1 - \bar{\tau}_{s,\rho})\,\bar{\tau}_{s,\rho}} &> \frac{\beta^2\kappa^2 - 2r_{eb}}{2 - \frac{1}{2}\beta^2\kappa^2 + \left(\frac{1}{2}\beta^2\kappa^2 - r_{eb}\right)\bar{\tau}_{s,\rho}} \\
&\iff & \left(\frac{2 - r_{eb}\bar{\tau}_{s,\rho}}{1 - \bar{\tau}_{s,\rho}} - \frac{1}{2}\beta^2\kappa^2\right)(1 - 2\bar{\tau}_{s,\rho}) &> \left(\beta^2\kappa^2 - 2r_{eb}\right)\bar{\tau}_{s,\rho} \\
&\iff & (2 - r_{eb}\bar{\tau}_{s,\rho})(1 - 2\bar{\tau}_{s,\rho}) &> \left(\frac{1}{2}\beta^2\kappa^2 - 2r_{eb}\bar{\tau}_{s,\rho}\right)(1 - \bar{\tau}_{s,\rho}) \\
&\iff & 2\,(1 - 2\bar{\tau}_{s,\rho}) &> \frac{1}{2}(1 - \bar{\tau}_{s,\rho})\,\beta^2\kappa^2 - \bar{\tau}_{s,\rho}r_{eb} \\
&\iff & 2 - \frac{1}{2}\beta^2\kappa^2 &> \left(4 - r_{eb} - \frac{1}{2}\beta^2\kappa^2\right)\bar{\tau}_{s,\rho}. \qquad (10)
\end{aligned}$$

Applying bounds to inequality (10) gives that $\partial\text{LHS}/\partial\bar{\tau}_{s,\rho} > \partial\text{RHS}/\partial\bar{\tau}_{s,\rho}$ when $\bar{\tau}_{s,\rho} < 3/10$, regardless of the other parameters in the model. Then when $e < 0$, κ is decreasing in $\bar{\tau}_{s,\rho}$ when $\bar{\tau}_{s,\rho} < 3/10$ and increasing when $\bar{\tau}_{s,\rho} > 1/2$. We now address the remaining gap, $\bar{\tau}_{s,\rho} \in [3/10, 1/2]$.

Note that inequality (10) becomes more difficult to satisfy as κ increases. In particular, the derivative of the left-hand side is $-\beta^2\kappa$ while the derivative of the right-hand side is

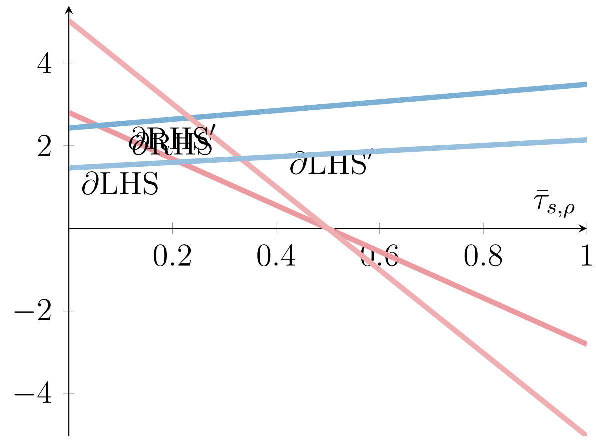

Figure 4: A graphical depiction of the comparative statics of κ with respect to $\bar{\tau}_{s,\rho}$ when $e < 0$ and $\bar{\tau}_{s,\rho} < 1/2$. This is just a doodle for intuition.

$-\beta^2\kappa\bar{\tau}_{s,\rho}$; since $\bar{\tau}_{s,\rho} < 1/2$, the derivative of the right-hand side is less negative (hence larger) than the derivative of the left-hand side. By earlier arguments, we know that there is a $\hat{\tau}^\star$ such that $\partial\text{LHS}/\partial\bar{\tau}_{s,\rho} = \partial\text{RHS}/\partial\bar{\tau}_{s,\rho}$. Consider the effect on κ of a small *decrease* in $\bar{\tau}_{s,\rho}$ from this point. Since $\bar{\tau}_{s,\rho}$ is moving into the range on which $\partial\text{LHS}/\partial\bar{\tau}_{s,\rho} > \partial\text{RHS}/\partial\bar{\tau}_{s,\rho}$, κ should intuitively increase (this direction is "backwards", since we are looking at a decrease in $\bar{\tau}_{s,\rho}$, and thus a negative derivative), ignoring fixed point effects. However, since $\partial\text{LHS}/\partial\bar{\tau}_{s,\rho} = \partial\text{RHS}/\partial\bar{\tau}_{s,\rho}$ at $\hat{\tau}^\star$, fixed point effects are relevant. If κ were to decrease following this small decrease in $\bar{\tau}_{s,\rho}$, inequality (10) becomes easier to satisfy. Then $\partial\text{LHS}/\partial\bar{\tau}_{s,\rho} > \partial\text{RHS}/\partial\bar{\tau}_{s,\rho}$ in both first- and second-order effects, implying that κ increases. This contradicts the assumption that κ decreases. Then a small decrease in $\bar{\tau}_{s,\rho}$ results in an increase in κ.

It then follows that from any $\bar{\tau}_{s,\rho}$ such that $\partial\text{LHS}/\partial\bar{\tau}_{s,\rho} = \partial\text{RHS}/\partial\bar{\tau}_{s,\rho}$, κ is *increasing* to the left of this intersection (we showed κ was decreasing with a decrease in $\bar{\tau}_{s,\rho}$, therefore κ is increasing in $\bar{\tau}_{s,\rho}$). Letting $\tau^\star$ be the maximum such $\hat{\tau}^\star$ completes the proof. □

Proof of Proposition 4

From Lemma 5, equilibrium first period prices must satisfy

$$p^c_{i,1} = \frac{1}{2b}\mathbb{E}\left[bc_i + a + ep^c_{j,1} + \frac{1}{2b}\left(a - bc_i + e\mathbb{E}\left[p^c_{j,2}\middle|\,\rho, s_\rho, p_1\right]\right)\frac{\partial}{\partial p_{i,1}}\mathbb{E}\left[p^\star_{j,2}\middle|\,\rho, s_\rho, p_1\right]\middle|\, s_\rho, s_{i,\theta}\right].$$

The expectation of second period price is

$$\mathbb{E}\left[\mathbb{E}\left[p^c_{j,2}\middle|\,\rho, s_\rho, p_1\right]\middle|\, s_{i,\theta}, s_\rho\right] = \frac{a}{2b-e} + \frac{\beta-1}{2}\mathbb{E}\left[\mathbb{E}[c_i|\rho, s_\rho, p_{i,1}]\middle|\, s_{i,\theta}, s_\rho\right] + \frac{be}{4b^2-e^2}\mathbb{E}\left[\mathbb{E}[c_j|\rho, s_\rho, p_{i,1}]\middle|\, s_{i,\theta}, s_\rho\right].$$

The conditional expectation of each firm's cost in the second period is

$$\mathbb{E}[c_k|\rho, s_\rho, p_{k,1}] = \rho + \mathbb{E}[\theta_k|s_{k,\theta}].$$

The expectation of this conditional expectation in the first period are

$$\begin{aligned}\mathbb{E}\left[\mathbb{E}[c_i|\rho, s_\rho, p_{i,1}]\right| s_{i,\theta}, s_\rho] &= \mathbb{E}[\rho|s_\rho] + \mathbb{E}[\theta_i|s_{i,\theta}] \\ \mathbb{E}\left[\mathbb{E}[c_j|\rho, s_\rho, p_{i,1}]\right| s_{i,\theta}, s_\rho] &= \mathbb{E}[\rho|s_\rho] + \mu_\theta\end{aligned}$$

Additionally, an increase in first period price effects the public expectation of cost in the second period directly by the coefficient on the signal.

$$\frac{\partial \mathbb{E}[c_i|\rho, s_\rho, p_{i,1}]}{\partial p_{i,1}} = \frac{\partial E[c_i|s_{i,\theta}]}{\partial s_{i,\theta}} \frac{\partial s_{i,\theta}}{\partial p_{i,1}} = \frac{\bar{\tau}_{i,\theta}}{\tilde{p}_\theta} = \frac{1}{p_\theta}$$

Given the simplified information structure, the first order condition in becomes

$$\begin{aligned}2bp^c_{i,1} &= b(\mathbb{E}[\theta_i|s_{i,\theta}] + \mathbb{E}[\rho|s_\rho]) + a + e\mathbb{E}[p^c_{j,1}|s_\rho] \\ &\quad + b\beta\frac{1}{p_{\theta,c}}\left[\frac{a}{2b-e} + \frac{\beta-1}{2}(\mathbb{E}[\theta_i|s_{i,\theta}] + E[\rho|s_\rho]) + \frac{be}{4b^2-e^2}\left(\mathbb{E}[\rho|s_\rho] + \mu_\theta\right)\right]\end{aligned}$$

Rearranging and matching coefficients, we have that any linear equilibrium satisfies the following system of equations.

$$\begin{aligned}p_{\theta,c}(1-2p_{\theta,c}) &= \frac{\beta(1-\beta)}{2} \\ p_{\theta,c}\left(1+\frac{e}{b}p_{\rho,c} - 2p_{\rho,c}\right) &= \left(\frac{1-\beta}{2} - \frac{be}{4b^2-e^2}\right)\beta \\ p_{\theta,c}\left(\frac{a}{b} + \frac{e}{b}(p_{0,c} + p_{\theta,c}\mu_\theta) - 2p_{0,c}\right) &= -\beta\left(\frac{a}{2b-e} + \frac{be\mu_\theta}{4b^2-e^2}\right)\end{aligned}$$

where $p^c_{i,1} = p_{0,c} + p_{\rho,c}E[\rho|s_\rho] + p_{\theta,c}E[\theta_i|s_{i,\theta}]$. The first equation has two solutions for $p_{\theta,c}$, but the second order condition only holds for $p_{\theta,c} = (1-\beta)/2$. Solving the remaining equations given this value for p_θ we get

$$\begin{aligned}
p_{\rho,c}(2b-e) &= b + \frac{2\beta b}{1-\beta}\left(\frac{\beta-1}{2} + \frac{be}{4b^2-e^2}\right) \\
\Rightarrow p_{\rho,c} &= \frac{b}{2b-e} + \frac{be}{4b^2-e^2}\beta \\
p_{0,c}(2b-e) &= a + \frac{(1-\beta)e\mu_\theta}{2} + \frac{2b\beta}{1-\beta}\left(\frac{a}{2b-e} + \frac{be}{4b^2-e^2}\mu_\theta\right) \\
\Rightarrow p_{0,c} &= \frac{2a+(1-\beta)e\mu_\theta}{2(2b-e)} + \frac{2b\beta}{1-\beta}\left(\frac{a}{2b-e} + \frac{be}{4b^2-e^2}\mu_\theta\right)
\end{aligned}$$

Because $\kappa \leq 2/(1-\beta)$ with the inequality strict for $e \neq 0$, then

$$p_\theta = \frac{1}{2+\beta\kappa} \geq \frac{1}{2+\frac{2\beta}{1-\beta}} = \frac{1-\beta}{2} = p_{\theta,c}$$

with the inequality strict when $e \neq 0$.

Defining $r = \frac{e}{b}$ we know that $r \in [-1,1]$ and the sign of r is the same as the sign of e. Then we can write both $p_{\rho,c}$ and p_ρ in terms of r,

$$p_{\rho,c} = \frac{1}{2-r} + \frac{r}{4-r^2}\beta \text{ and } p_\rho = \frac{\frac{(2-r)-(1-r)\beta\kappa}{2-r}}{2 - r\bar{\tau}_{s,\rho} - \frac{1}{2}\beta^2\kappa^2(1-\bar{\tau}_{s,\rho})}.$$

Becuase $\beta\kappa \leq \frac{e}{b} \leq 1$ then p_ρ can be bounded as follows when $r \geq 0$

$$p_\rho \leq \frac{(2-r)-(1-r)\beta\kappa}{(2-r)^2} \leq \frac{1}{2-r} \leq p_{\rho,c}.$$

Proof of Proposition 5

Comparing the first order conditions in Lemma 3 and Lemma 5, all terms are the same except $\mathbb{E}[\frac{\partial}{\partial p_{i,1}}\mathbb{E}\left[p_{j,2}|\,\rho, p_{i,1}, s\right]|s]$ and $\mathbb{E}[\mathbb{E}\left[p_{j,2}|\,\rho, p_{i,1}, s\right]|s]$. From Proposition 4 we know that the first term is constant for all signals, and is larger when firms share common cost information than when they do not. Taking the expectation of the second term it is the same in either case.

$$\begin{aligned}
&\mathbb{E}[\mathbb{E}[\mathbb{E}\left[p_{j,2}|\,\rho, p_{i,1}, s\right]|s]] \\
&= \frac{1}{4b^2-e^2}\Big(\mathbb{E}[(2b+e)a + (2b^2+be)(\mu_\theta + E[\rho|s_{i,\rho}])] + be\mathbb{E}[\kappa(\mathbb{E}[p_{i,1}|s] - p_0 + p_\theta\mu_\theta + p_\rho\mathbb{E}[\rho|s])]\Big)
\end{aligned}$$

The later term equals zero in both the sharing and non-sharing settings as $\mathbb{E}[\mathbb{E}[p_{i,1}|s] -$

$p_0+p_\theta\mu_\theta+p_\rho E[\rho|s])] = 0$. Moreover, the former term only consists of constants and expected costs which are the same in each case. Therefore

$$\mathbb{E}[p_{i,2}] = \mathbb{E}[p^c_{i,2}] = \frac{a + b(\mu_\rho + \mu_\theta)}{2b - e}$$

Imposing symmetry and taking the expectation of optimal pricing equation we have,

$$\mathbb{E}[2bp_{i,1}] = \mathbb{E}\left[\mathbb{E}\left[bc_i + a + ep_{i,1} + \frac{1}{2b}\left(a - bc_i + e\mathbb{E}\left[p^c_{j,2}\middle|\,\rho, s_\rho, p_1\right]\right)\frac{be}{4b^2 - e^2}\kappa\middle|\, s_\rho, s_{i,\theta}\right]\right].$$

where all terms are the same except κ, which is larger and denoted by κ^c in the case where firms share common cost information. Then

$$\begin{aligned}\mathbb{E}[p_{i,1}] &= \frac{1}{2b-e}\left(a + b(\mu_\rho + \mu_\theta) + \frac{e\kappa}{2(4b^2 - e^2)}\mathbb{E}\left[\mathbb{E}\left[\left(a - bc_i + e\mathbb{E}\left[p_{j,2}|\,\rho, s_\rho, p_1\right]\right)|\, s_\rho, s_{i,\theta}\right]\right]\right)\\ &\le \frac{1}{2b-e}\left(a + b(\mu_\rho + \mu_\theta) + \frac{e\kappa^c}{2(4b^2 - e^2)}\mathbb{E}\left[\mathbb{E}\left[\left(a - bc_i + e\mathbb{E}\left[p^c_{j,2}\middle|\,\rho, s_\rho, p_1\right]\right)\middle|\, s_\rho, s_{i,\theta}\right]\right]\right)\\ &= \mathbb{E}[p^c_{i,1}]\end{aligned}$$

Proof of Lemma 6

Ex ante expected profits in the first period are given by

$$\begin{aligned}\mathbb{E}\left[\mathbb{E}\left[\pi_{i,1}|\, s_i\right]\right] &= \mathbb{E}\left[\mathbb{E}\left[(a - bp_i + ep_j)(p_i - c_i)|\, s_i\right]\right]\\ &= \mathbb{E}\left[\mathbb{E}\left[ap_i - ac_i - bp_i^2 + bc_ip_i + ep_ip_j - ec_ip_j\middle|\, s_i\right]\right].\end{aligned}$$

Any terms which contain only first powers of uncorrelated variables are invariant to the investment in precision, and can be subsumed into an additional parameter. For example,

$$\begin{aligned}\mathbb{E}\left[\mathbb{E}\left[ap_i|\, s_i\right]\right] &= \mathbb{E}\left[\mathbb{E}\left[ap_{i0} + ap_{i\theta}\mathbb{E}\left[\theta_i|\, s_{i,\theta}\right] + ap_{i\rho}\mathbb{E}\left[\rho|\, s_{i,\rho}\right]|\, s_i\right]\right]\\ &= ap_{i0} + ap_{i\theta}\mathbb{E}\left[\mathbb{E}\left[\theta_i|\, s_i\right]\right] + ap_{i\rho}\mathbb{E}\left[\mathbb{E}\left[\rho|\, s_i\right]\right]\\ &= ap_{i0} + ap_{i\theta}\mathbb{E}\left[\theta_i\right] + ap_{i\rho}\mathbb{E}\left[\rho\right] = ap_{i0} + ap_{i\theta}\mu_\theta + ap_{i\rho}\mu_\rho;\\ \mathbb{E}\left[\mathbb{E}\left[ac_i|\, s_i\right]\right] &= \mathbb{E}\left[\mathbb{E}\left[(\theta_i + \rho)\, a|\, s_i\right]\right]\\ &= a\mathbb{E}\left[\mathbb{E}\left[\theta_i + \rho|\, s_i\right]\right] = a\theta_i + a\rho.\end{aligned}$$

Lemma 15. *The interrelations of the conditional expected values of cost components are as* *follows:*

$$\mathbb{E}\left[\mathbb{E}\left[\theta_i|\, s_i\right]^2\right] = \frac{\tau_{i,\theta}}{\left(\tau_{i,\theta}+\tau_\theta\right)\tau_\theta} + \mu_\theta^2,$$
$$\mathbb{E}\left[\mathbb{E}\left[\rho|\, s_i\right]^2\right] = \frac{\tau_{i,\rho}}{\left(\tau_{i,\rho}+\tau_\rho\right)\tau_\rho} + \mu_\rho^2,$$
$$\mathbb{E}\left[\mathbb{E}\left[\rho|\, s_i\right]\mathbb{E}\left[\rho|\, s_j\right]\right] = \frac{\tau_{i,\rho}\tau_{j,\rho}}{\left(\tau_{i,\rho}+\tau_\rho\right)\left(\tau_{j,\rho}+\tau_\rho\right)\tau_\rho} + \mu_\rho^2.$$

Then first-period profits rely on the investment in precision only through the last four parameters.

$$\begin{aligned}
\mathbb{E}\left[\mathbb{E}\left[p_i^2\middle|\, s_i\right]\right] &= \mathbb{E}\left[\mathbb{E}\left[\left(p_{i0}+p_{i\theta}\mathbb{E}\left[\theta_i|\, s_i\right]+p_{i\rho}\mathbb{E}\left[\rho|\, s_i\right]\right)^2\middle|\, s_i\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[p_{i0}^2+2p_{i0}p_{i\theta}\mathbb{E}\left[\theta_i|\, s_i\right]+2p_{i0}p_{i\rho}\mathbb{E}\left[\rho|\, s_i\right]\middle|\, s_i\right]\right] \\
&\quad + \mathbb{E}\left[\mathbb{E}\left[p_{i\theta}^2\mathbb{E}\left[\theta_i|\, s_i\right]^2+2p_{i\theta}p_{i\rho}\mathbb{E}\left[\theta_i|\, s_i\right]\mathbb{E}\left[\rho|\, s_i\right]+p_{i\rho}^2\mathbb{E}\left[\rho|\, s_i\right]^2\middle|\, s_i\right]\right] \\
&= \frac{\tau_{i,\theta}p_{i\theta}^2}{\left(\tau_{i,\theta}+\tau_\theta\right)\tau_\theta} + \frac{\tau_{i,\rho}p_{i\rho}^2}{\left(\tau_{i,\rho}+\tau_\rho\right)\tau_\rho} + C_{i1}.
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[\mathbb{E}\left[c_ip_i|\, s_i\right]\right] &= \mathbb{E}\left[\mathbb{E}\left[\left(\theta_i+\rho\right)\left(p_{i0}+p_{i\theta}\mathbb{E}\left[\theta_i|\, s_i\right]+p_{i\rho}\mathbb{E}\left[\rho|\, s_i\right]\right)\middle|\, s_i\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[p_{i\theta}\theta_i\mathbb{E}\left[\theta_i|\, s_i\right]+p_{i\rho}\rho\mathbb{E}\left[\rho|\, s_i\right]\middle|\, s_i\right]\right] + C_{i21} \\
&= \mathbb{E}\left[p_{i\theta}\mathbb{E}\left[\theta_i|\, s_i\right]^2+p_{i\rho}\mathbb{E}\left[\rho|\, s_i\right]^2\right] + C_{i21} \\
&= \frac{\tau_{i,\theta}p_{i\theta}}{\left(\tau_{i,\theta}+\tau_\theta\right)\tau_\theta} + \frac{\tau_{i,\rho}p_{i\rho}}{\left(\tau_{i,\rho}+\tau_\rho\right)\tau_\rho} + C_{i2}.
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[\mathbb{E}\left[p_ip_j|\, s_i\right]\right] &= \mathbb{E}\left[\mathbb{E}\left[p_ip_j|\, s_i,s_j\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left(p_{i0}+p_{i\theta}\mathbb{E}\left[\theta_i|\, s_i\right]+p_{i\rho}\mathbb{E}\left[\rho|\, s_i\right]\right)\left(p_{j0}+p_{j\theta}\mathbb{E}\left[\theta_j|\, s_j\right]+p_{j\rho}\mathbb{E}\left[\rho|\, s_j\right]\right)\middle|\, s_i,s_j\right]\right] \\
&= \mathbb{E}\left[p_{i\rho}p_{j\rho}\mathbb{E}\left[\rho|\, s_i\right]\mathbb{E}\left[\rho|\, s_j\right]\right] + C_{i31} \\
&= \frac{\tau_{i,\rho}\tau_{j,\rho}p_{i\rho}p_{j\rho}}{\left(\tau_{i,\rho}+\tau_\rho\right)\left(\tau_{j,\rho}+\tau_\rho\right)\tau_\rho} + C_{i3}.
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[\mathbb{E}\left[c_ip_j|\, s_i\right]\right] &= \mathbb{E}\left[\mathbb{E}\left[c_ip_j|\, s_i,s_j\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left(\theta_i+\rho\right)\left(p_{j0}+p_{j\theta}\mathbb{E}\left[\theta_j|\, s_j\right]+p_{j\rho}\mathbb{E}\left[\rho|\, s_j\right]\right)\middle|\, s_i,s_j\right]\right] \\
&= \mathbb{E}\left[p_{j\rho}\mathbb{E}\left[\rho|\, s_i\right]\mathbb{E}\left[\rho|\, s_j\right]\right] + C_{i41} \\
&= \frac{\tau_{i,\rho}\tau_{j,\rho}p_{j\rho}}{\left(\tau_{i,\rho}+\tau_\rho\right)\left(\tau_{j,\rho}+\tau_\rho\right)\tau_\rho} + C_{i4}.
\end{aligned}$$

Putting everything together, we are left with

$$\begin{aligned}\mathbb{E}\left[\pi_{i,1}\right] &= -\left(\frac{\tau_{i,\theta}p_{i\theta}^2}{\left(\tau_{i,\theta}+\tau_\theta\right)\tau_\theta}+\frac{\tau_{i,\rho}p_{i\rho}^2}{\left(\tau_{i,\rho}+\tau_\rho\right)\tau_\rho}\right)b+\left(\frac{\tau_{i,\theta}p_{i\theta}}{\left(\tau_{i,\theta}+\tau_\theta\right)\tau_\theta}+\frac{\tau_{i,\rho}p_{i\rho}}{\left(\tau_{i,\rho}+\tau_\rho\right)\tau_\rho}\right)b \\ &\quad+\left(\frac{\tau_{i,\rho}\tau_{j,\rho}p_{i\rho}p_{j\rho}}{\left(\tau_{i,\rho}+\tau_\rho\right)\left(\tau_{j,\rho}+\tau_\rho\right)\tau_\rho}\right)e-\left(\frac{\tau_{i,\rho}\tau_{j,\rho}p_{j\rho}}{\left(\tau_{i,\rho}+\tau_\rho\right)\left(\tau_{j,\rho}+\tau_\rho\right)\tau_\rho}\right)e+C_i \\ &= \left(\frac{\left(1-p_{i\theta}\right)p_{i\theta}\tau_{i,\theta}}{\left(\tau_{i,\theta}+\tau_\theta\right)\tau_\theta}\right)b+\left(\frac{\left(1-p_{i\rho}\right)p_{i\rho}\tau_{i,\rho}}{\left(\tau_{i,\rho}+\tau_\rho\right)\tau_\rho}\right)b-\left(\frac{\left(1-p_{i\rho}\right)p_{j\rho}\tau_{i,\rho}\tau_{j,\rho}}{\left(\tau_{i,\rho}+\tau_\rho\right)\left(\tau_{j,\rho}+\tau_\rho\right)\tau_\rho}\right)e+C_i.\end{aligned}$$

Proof of Lemma 8 First period profits for given realization of signals $(s_\rho, s_{i,\theta})$ are

$$\begin{aligned}\pi_{i,1}(s_\rho, s_{i,\theta}) &= (a+e\mathbb{E}[p_{j,1}(s_\rho, s_{j,\theta})|s_\rho])(p_{i,1}^c(s_\rho, s_{i,\theta})-\mathbb{E}[c_i|s_\rho, s_{i,\theta}]) \\ &\quad - b(p_{i,1}^c(s_\rho, s_{i,\theta}))^2+bp_{i,1}^c(s_\rho, s_{i,\theta})\mathbb{E}[c_i|s_\rho, s_{i,\theta}]\end{aligned}$$

Expected first period payoffs for given information precision levels[18] are

$$\begin{aligned}\mathbb{E}[\pi_{i,1}(s_p, s_{i,\theta})|\tau_{\varepsilon,\rho}, \tau_{\varepsilon,\theta}^i] &= a\mathbb{E}[p_{i,1}^c(s_\rho, s_{i,\theta})-\mathbb{E}[c_i|s_\rho, s_{i,\theta}]|\tau_{\varepsilon,\rho}, \tau_{\varepsilon,\theta}^i] \\ &\quad + e\mathbb{E}[\mathbb{E}[p_{j,1}(s_\rho, s_{j,\theta})|s_\rho](p_{i,1}^c(s_\rho, s_{i,\theta})-\mathbb{E}[c_i|s_\rho, s_{i,\theta}])|\tau_{\varepsilon,\rho}, \tau_{\varepsilon,\theta}^i] \\ &\quad - b\mathbb{E}[(p_{i,1}^c(s_\rho, s_{i,\theta}))^2|\tau_{\varepsilon,\rho}, \tau_{\varepsilon,\theta}^i]+b\mathbb{E}[p_{i,1}^c(s_\rho, s_{i,\theta})E[c_i|s_\rho, s_{i,\theta}]|\tau_{\varepsilon,\rho}, \tau_{\varepsilon,\theta}^i]\end{aligned}$$

Grouping together the terms that do not depend on precision into C this ex-ante first period payoff becomes

$$\begin{aligned}\mathbb{E}[\pi_{i,1}(s_p, s_{i,\theta})|\tau_{\varepsilon,\rho}, \tau_{\varepsilon,\theta}^i] &= e(p_\rho^2-p_\rho)\mathbb{E}[\mathbb{E}[\rho|s_\rho]|\tau_{\varepsilon,\rho}]+b(p_\rho-p_\rho^2)\mathbb{E}[(\mathbb{E}[\rho|s_\rho])^2|\tau_{\varepsilon,\rho}] \\ &\quad + b(p_\theta^2-p_\theta)\mathbb{E}[(\mathbb{E}[\theta_i|s_{i,\theta}])^2|\tau_{\varepsilon,\theta}^i]+C \\ &= b(p_\theta-p_\theta^2)\mathrm{Var}(\mathbb{E}[\theta_i|s_{i,\theta}])+(b-e)(p_\rho-p_\rho^2)\mathrm{Var}(\mathbb{E}[\rho|s_\rho])+C \\ &= b(p_\theta-p_\theta^2)\frac{\tau_{\varepsilon,\theta}^i}{(\tau_\theta+\tau_{\varepsilon,\theta}^i)\tau_\theta}+(b-e)(p_\rho-p_\rho^2)\frac{\tau_{\varepsilon,\rho}}{(\tau_\rho+\tau_{\varepsilon,\rho})\tau_\rho}+C\end{aligned}$$

Proof of Proposition 6

Proof. We have already seen (in Proposition 1) that conditional on precision there is a unique equilibrium in the pricing game. With information acquisition, opponent precision is unobservable. Fixing believed equilibrium inference $\hat{\kappa}$, it is the case that the firm's pricing *strategy* is independent of its level of precision investment: prices are functions of the

[18]Firm i's payoffs are not affected by the precision of information firm j has about its private cost component

conditional expectations of cost parameters, and the entire effect of precision is captured in the conditional expectation. Lemma 6 implies that there are strictly decreasing marginal returns to precision, and since the cost of precision is convex it follows that there is a unique level of precision, and hence unique pricing strategy, taking as given believed equilibrium inference $\hat{\kappa}$.

Then equilibrium will be nonunique only if there are two equilibrium inference levels $\hat{\kappa}$ and $\hat{\kappa}'$, leading to distinct precision and pricing decisions. Suppose $\hat{\kappa}$ is an equilibrium belief, and $\hat{\kappa}' > \hat{\kappa}$. By Proposition 3, $p'_\theta < p_\theta$. Since $p_\theta < 1/2$ in any equilibrium, Lemma 7 implies that the marginal return to precision is lower under belief $\hat{\kappa}'$ than under belief $\hat{\kappa}$. Then $\tau'_{i,\theta} < \tau_{i,\theta}$. Then Proposition 3 implies that $\hat{\kappa}' < \hat{\kappa}$, a contradiction. Then there is a unique level of informativeness $\hat{\kappa}$ and unique choice of $\tau_{i,\theta}$. □

B Conditional expectations

The following expectations are used throughout the paper.

$$\mathbb{E}[x|s_{i,x}] = s_{i,\rho}\bar{\tau}_{s,x,i} + \mu_x(1 - \bar{\tau}_{s,x,i}) \tag{11}$$

$$\text{Var}\left(\mathbb{E}[x|s_{i,x}]\right) = \bar{\tau}^2_{s,x,i}(\text{Var}(x|\bar{\tau}) + \text{Var}(\varepsilon_x|\bar{\tau})) = \frac{\bar{\tau}_{s,x,i}}{\tau_x} \tag{12}$$

$$\mathbb{E}\left[p_{j,1}|\, s_{i,\rho}\right] = p_{0,j} + \mu_\theta p_{\theta,j} + \bar{\tau}_{s,\rho,j}\mathbb{E}\left[\rho|\, s_{i,\rho}\right] p_{\rho,j} + \left(1 - \bar{\tau}_{s,\rho,j}\right)\mu_\rho p_{\rho,j} \tag{13}$$

$$\text{Var}(p_1|\bar{\tau}) = p^2_\theta(\bar{\tau})\text{Var}\left(\mathbb{E}[\theta|s_\theta]|\bar{\tau}\right) + p^2_\rho(\bar{\tau})\text{Var}\left(\mathbb{E}[\rho|s_\rho]|\bar{\tau}\right) \tag{14}$$

$$\text{Cov}(p_{i,1}, p_{j,1}|\bar{\tau}) = p^2_\rho(\bar{\tau})\text{Cov}\left(\mathbb{E}[\rho|s_{i,\rho}], \mathbb{E}[\rho|s_{j,\rho}]|\bar{\tau}\right) \tag{15}$$

$$\mathbb{E}[x|s_{i,x}, s_{j,x}] = \frac{\tau_{s,x}}{\tau_x + 2\tau_{s,x}}(s_{i,x} + s_{j,x}) + \left(1 - \frac{2\tau_{s,x}}{\tau_x + 2\tau_{s,x}}\right)\mu_x \tag{16}$$

$$\text{Var}\left(\mathbb{E}[\rho|s_{i,\rho}, s_{j,\rho}]\right) = \frac{2\tau_{s,\rho,i}}{\tau_\rho(\tau_\rho + 2\tau_{s,\rho})} \in \left(\frac{\bar{\tau}_{s,\rho,i}}{\tau_\rho}, \frac{2\bar{\tau}_{s,\rho,i}}{\tau_\rho}\right) \tag{17}$$

$$\text{Cov}\left(\mathbb{E}[\rho|s_{i,\rho}], \mathbb{E}[\rho|s_{j,\rho}]|\bar{\tau}_{s,\rho,i}\right) = \frac{\bar{\tau}^2_{s,\rho,i}}{\tau_\rho} < \frac{1}{2}\text{Var}\left(\mathbb{E}[\rho|s_{i,\rho}, s_{j,\rho}]\right) \tag{18}$$

Second period expectations

$$
\begin{aligned}
\mathbb{E}[p_{i,2}|\bar{\tau}_{s,i}] &= \frac{a + b(\mu_\rho + \mu_\theta)}{2b - e} && (19)\\
\text{Var}(p_{i,2}|\bar{\tau}) &= \frac{1}{4}\text{Var}(c_i) + \frac{e^2}{4b^2}\text{Var}\left(\mathbb{E}[p_{j,2}|\rho, p_{i,1}, p_{j,1}]|\bar{\tau}\right)\\
&\quad - \frac{e}{4b}\text{Cov}\left(c_i, \mathbb{E}[p_{j,1}|\rho, p_{i,1}, p_{j,1}]|\bar{\tau}\right) && (20)\\
\text{Cov}(p_{i,2}, p_{j,2}|\bar{\tau}) &= \frac{1}{4}\text{Cov}(c_i, c_j|\bar{\tau}) + \frac{e}{2b}\text{Cov}\left(c_i, \mathbb{E}[p_{j,2}|\rho, p_{i,1}, p_{j,1}]|\bar{\tau}\right)\\
&\quad + \frac{e^2}{4b^2}\text{Cov}\left(\mathbb{E}[p_{i,2}|\rho, p_{i,1}, p_{j,1}], \mathbb{E}[p_{j,2}|\rho, p_{i,1}, p_{j,1}]|\bar{\tau}\right) && (21)
\end{aligned}
$$

Additional useful expectations

$$
\begin{aligned}
\text{Var}\left(\mathbb{E}[p_{j,2}|\rho, p_{i,1}, p_{j,1}]|\bar{\tau}\right) &= \frac{1}{(4b^2 - e^2)^2}\Big\{4b^4\text{Var}\left(\mathbb{E}[c_j|\rho, p_{j,1}]|\bar{\tau}\right) + b^2e^2\text{Var}\left(\mathbb{E}[c_i|\rho, p_{i,1}]|\bar{\tau}\right)\\
&\quad -2(2b^2)(be)\text{Cov}\left(\mathbb{E}[c_j|\rho, p_{j,1}], \mathbb{E}[c_i|\rho, p_{i,1}]|\bar{\tau}\right)\Big\} && (22)\\
\text{Cov}\left(\mathbb{E}[p_{i,2}|\rho, p_1], \mathbb{E}[p_{j,2}|\rho, p_1]|\bar{\tau}\right) &= \frac{4b^4 + b^2e^2}{(4b^2 - e^2)^2}\text{Cov}\left(\mathbb{E}[c_j|\rho, p_{j,i}], \mathbb{E}[c_i|\rho, p_{i,1}]|\bar{\tau}\right)\\
&\quad + \frac{4b^3e}{(4b^2 - e^2)^2}\text{Var}\left(\mathbb{E}[c_i|\rho, p_{i,1}]|\bar{\tau}\right) && (23)\\
\text{Var}\left(\mathbb{E}[c_i|\rho, p_{i,1}]|\bar{\tau}\right) &= \text{Var}(\rho|\bar{\tau}) + \kappa^2\bar{\tau}^2_{s,\rho,i}p^2_\rho(\bar{\tau})\text{Var}(\varepsilon_{i,\rho}|\bar{\tau}) + \kappa^2p^2_\theta(\bar{\tau})\text{Var}\left(\mathbb{E}[\theta_i|s_{i,\theta}]|\bar{\tau}\right) && (24)\\
\text{Cov}\left(\mathbb{E}[c_j|\rho, p_{j,i}], \mathbb{E}[c_i|\rho, p_{i,1}]|\bar{\tau}\right) &= \text{Var}(\rho|\bar{\tau}) && (25)\\
\text{Cov}(\rho, p_{i,1}|\bar{\tau}) &= p_\rho(\bar{\tau}_{s,\rho,i})\bar{\tau}_{s,\rho,i}\text{Var}(\rho|\bar{\tau}) && (26)\\
\text{Cov}\left(c_i, \mathbb{E}[p_{j,2}|\rho, p_{i,1}, p_{j,1}]|\bar{\tau}\right) &= \frac{2b^2}{4b^2 - e^2}\text{Cov}\left(c_i, \mathbb{E}[c_j|\rho, p_{j,1}]|\bar{\tau}\right) + \frac{be}{4b^2 - e^2}\text{Cov}\left(c_i, \mathbb{E}[c_i|\rho, p_{i,1}]|\bar{\tau}\right) && (27)\\
\text{Cov}\left(c_i, \mathbb{E}[c_j|\rho, p_{j,1}]|\bar{\tau}\right) &= \text{Var}(\rho|\bar{\tau}) && (28)
\end{aligned}
$$

$$\text{Cov}\left(c_i, \mathbb{E}[c_i|\rho, p_{i,1}]|\bar{\tau}\right) = \text{Var}(\rho|\bar{\tau}) + \kappa\bar{\tau}_{s,\theta,i}p_\theta(\bar{\tau})\text{Var}(\theta_i|\bar{\tau}) \tag{29}$$

$$\begin{aligned} \text{Var}\left(\mathbb{E}[p^c_{j,2}|\rho, p^c_{i,1}, p^c_{j,1}]|\bar{\tau}\right) &= \frac{1}{(4b^2-e^2)^2}\left\{4b^4\left(\text{Var}(\rho|\bar{\tau}) + \text{Var}\left(\mathbb{E}[\theta_j|s_{j,\theta}]|\bar{\tau}\right)\right)\right. \\ &\left. +b^2e^2\left(\text{Var}(\rho|\bar{\tau}) + \text{Var}\left(\mathbb{E}[\theta_i|s_{i,\theta}]|\bar{\tau}\right)\right) - 4b^3e\text{Var}(\rho|\bar{\tau})\right\} \end{aligned} \tag{30}$$

$$\text{Cov}\left(c_i, \mathbb{E}[p^c_{j,2}|\rho, p^c_{i,1}, p^c_{j,1}]|\bar{\tau}\right) = \frac{1}{4b^2-e^2}\left(2b^2\text{Var}(\rho|\bar{\tau}) + be\left(\text{Var}(\rho|\bar{\tau}) + \text{Var}\left(\mathbb{E}[\theta_i|s_{i,\theta}]|\bar{\tau}\right)\right)\right) \tag{31}$$

$$\begin{aligned} \text{Cov}\left(\mathbb{E}[p^c_{i,2}|\cdot], \mathbb{E}[p^c_{j,2}|\cdot]|\bar{\tau}\right) &= \frac{4b^2e^2+e^4}{4(4b^2-e^2)^2}\text{Cov}\left(\mathbb{E}[c_j|\rho, p^c_{j,i}], \mathbb{E}[c_i|\rho, p^c_{i,1}]|\bar{\tau}\right) \\ &+ \frac{be^3}{(4b^2-e^2)^2}\text{Var}\left(\mathbb{E}[c_i|\rho, p^c_{i,1}]|\bar{\tau}\right) \end{aligned} \tag{32}$$

B.1 Proofs of expectations

Proof of expectation (11).

$$\begin{aligned} \mathbb{E}[x|s_{i,x}] &= \mu_x + \frac{\sigma_x^2}{\sigma_x^2+\sigma_{s,\rho}^2}(s_{i,x}-\mu_x) \\ &= s_{i,\rho}\bar{\tau}_{s,x,i} + \mu_x(1-\bar{\tau}_{s,x,i}) \end{aligned}$$

□

Proof of expectation (12).

$$\begin{aligned} \text{Var}\left(\mathbb{E}[x|s_{i,x}]\right) &= \text{Var}\left(s_{i,\rho}\bar{\tau}_{s,x,i} + \mu_x(1-\bar{\tau}_{s,x,i})\right) \\ &= \bar{\tau}^2_{s,x,i}\left(\sigma^2_{s,x}+\sigma^2_x\right) \\ &= \frac{\bar{\tau}_{s,x,i}}{\tau_x} \end{aligned}$$

□

Proof of expectation (13).

$$\begin{aligned} \mathbb{E}\left[p_{j,1}|\,s_{i,\rho}\right] &= \mathbb{E}\left[p_{0,j} + \mathbb{E}\left[\theta_j|\,s_{j,\theta}\right]p_{\theta,j} + \mathbb{E}\left[\rho|\,s_{j,\rho}\right]p_{\rho,j}|\,s_{i,\rho}\right] \\ &= p_{0,j} + \mathbb{E}\left[\bar{\tau}_{s,\theta,j}s_{j,\theta} + (1-\bar{\tau}_{s,\theta,j})\,\mu_\theta|\,s_{i,\rho}\right]p_{\theta,j} + \mathbb{E}\left[\bar{\tau}_{s,\rho,j}s_{j,\rho} + (1-\bar{\tau}_{s,\rho,j})\,\mu_\rho|\,s_{i,\rho}\right] \\ &= p_{0,j} + \mu_\theta p_{\theta,j} + \bar{\tau}_{s,\rho,j}\mathbb{E}\left[\rho|\,s_{i,\rho}\right]p_{\rho,j} + (1-\bar{\tau}_{s,\rho,j})\,\mu_\rho p_{\rho,j} \end{aligned}$$

□

Proof of expectation (14).

$$
\begin{aligned}
\mathrm{Var}(p_1|\bar{\tau}) &= \mathbb{E}[(p_1 - \mathbb{E}[p_1|\bar{\tau}])^2|\bar{\tau}] \\
&= \mathbb{E}\left[\left(p_\theta \left(\mathbb{E}[\theta|s_\theta] - \mu_\theta\right) + p_\rho \left(\mathbb{E}[\rho|s_\rho] - \mu_\rho\right)\right)^2 \middle| \bar{\tau}\right] \\
&= \mathbb{E}\left[p_\theta^2 \mathrm{Var}\left(\mathbb{E}[\theta|s_\theta]\right) + p_\rho^2 \mathrm{Var}\left(\mathbb{E}[\rho|s_\rho]\right) + 2p_\theta p_\rho \mathrm{Cov}\left(\mathbb{E}[\theta|s_\theta], \mathbb{E}[\rho|s_\rho]\right) \middle| \bar{\tau}\right] \\
&= p_\theta^2(\bar{\tau})\mathrm{Var}\left(\mathbb{E}[\theta|s_\theta]|\bar{\tau}\right) + p_\rho^2(\bar{\tau})\mathrm{Var}\left(\mathbb{E}[\rho|s_\rho]|\bar{\tau}\right)
\end{aligned}
$$

□

Proof of expectation (15).

$$
\begin{aligned}
\mathrm{Cov}(p_{i,1}, p_{j,1}|\bar{\tau}) =& \mathbb{E}[(p_{i,1} - \mathbb{E}[p_{i,1}|\bar{\tau}])(p_{j,1} - \mathbb{E}[p_{j,1}|\bar{\tau}])|\bar{\tau}] \\
=& \mathbb{E}\left[p_\theta^2 \mathrm{Cov}\left(\mathbb{E}[\rho|s_{i,\rho}], \mathbb{E}[\rho|s_{j,\rho}]\right) + p_\rho^2 \mathrm{Cov}\left(\mathbb{E}[\rho|s_{i,\rho}], \mathbb{E}[\rho|s_{j,\rho}]\right)\right. \\
& \left. + p_\theta p_\rho \left(\mathrm{Cov}\left(\mathbb{E}[\rho|s_{i,\rho}], \mathbb{E}[\rho|s_{j,\rho}]\right) + \mathrm{Cov}\left(\mathbb{E}[\rho|s_{i,\rho}], \mathbb{E}[\rho|s_{j,\rho}]\right)\right) |\bar{\tau}\right] \\
=& p_\rho^2(\bar{\tau})\mathrm{Cov}\left(\mathbb{E}[\rho|s_{i,\rho}], \mathbb{E}[\rho|s_{j,\rho}]|\bar{\tau}\right)
\end{aligned}
$$

□

Proof of expectation (16). The variables $(x, s_{i,x}, s_{j,x})$ for $x = \rho, \theta$ are jointly distributed

$$
\begin{pmatrix} x \\ s_{i,x} \\ s_{j,x} \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_x \\ \mu_x \\ \mu_x \end{pmatrix}, \begin{pmatrix} \sigma_x^2 & \sigma_x^2 & \sigma_x^2 \\ \sigma_x^2 & \sigma_x^2 + \sigma_{s,x,i}^2 & \sigma_x^2 \\ \sigma_x^2 & \sigma_x^2 & \sigma_x^2 + \sigma_{s,x,j}^2 \end{pmatrix}\right).
$$

$$
\begin{aligned}
\mathbb{E}[x|s_{i,x}, s_{j,x}] &= \mu_x + \Sigma_{21}\Sigma_{22}^{-1}\begin{pmatrix} s_{i,x} - \mu_x \\ s_{j,x} - \mu_x \end{pmatrix} \\
\text{where, } \Sigma_{21} &= (\sigma_x^2 \ \sigma_x^2), \quad \Sigma_{22}^{-1} = \frac{1}{(\sigma_x^2 + \sigma_{s,x,i}^2)(\sigma_x^2 + \sigma_{s,x,j}^2) - \sigma_x^4}\begin{pmatrix} \sigma_x^2 + \sigma_{s,x,j}^2 & -\sigma_x^2 \\ -\sigma_x^2 & \sigma_x^2 + \sigma_{s,x,i}^2 \end{pmatrix} \\
\mathbb{E}[x|s_{i,x}, s_{j,x}] &= \frac{\sigma_{s,x,i}^2\sigma_{s,x,j}^2}{\sigma_x^2(\sigma_{s,x,i}^2 + \sigma_{s,x,j}^2) + \sigma_{s,x,i}^2\sigma_{s,x,j}^2}\mu_\rho + \frac{\sigma_x^2\sigma_{s,x,j}^2}{\sigma_x^2(\sigma_{s,x,i}^2 + \sigma_{s,x,j}^2) + \sigma_{s,x,i}^2\sigma_{s,x,j}^2}s_{i,\rho} \\
&\quad + \frac{\sigma_x^2\sigma_{s,x,i}^2}{\sigma_x^2(\sigma_{s,x,i}^2 + \sigma_{s,x,j}^2) + \sigma_{s,x,i}^2\sigma_{s,x,j}^2}s_{j,\rho}
\end{aligned}
$$

When $\sigma_{s,x,i}^2 = \sigma_{s,x,j}^2$, then this expression simplifies to

$$
\mathbb{E}[x|s_{i,x}, s_{j,x}] = \frac{\tau_{s,x}}{\tau_x + 2\tau_{s,x}}(s_{i,x} + s_{j,x}) + \left(1 - \frac{2\tau_{s,x}}{\tau_x + 2\tau_{s,x}}\right)\mu_x
$$

□

Proof of expectation (17).

$$\begin{aligned}
\mathrm{Var}\left(\mathbb{E}[\rho|s_{i,\rho}, s_{j,\rho}]\right) &= \mathrm{Var}\left(\frac{\tau_{s,\rho}}{\tau_\rho + 2\tau_{s,\rho}}(s_{i,\rho} + s_{j,\rho}) + \left(1 - \frac{2\tau_{s,\rho}}{\tau_\rho + 2\tau_{s,\rho}}\right)\mu_\rho\right) \\
&= \left(\frac{\tau_{s,\rho}}{\tau_\rho + 2\tau_{s,\rho}}\right)^2 \mathrm{Var}(s_{i,\rho} + s_{j,\rho}) \\
&= \left(\frac{\tau_{s,\rho}}{\tau_\rho + 2\tau_{s,\rho}}\right)^2 (4\mathrm{Var}(\rho) + \mathrm{Var}(\varepsilon_{i,\rho}) + \mathrm{Var}(\varepsilon_{j,\rho})) \\
&= \frac{2\tau_{s,\rho}}{\tau_\rho(\tau_\rho + 2\tau_{s,\rho})}
\end{aligned}$$

□

Proof of expectation (18).

$$\begin{aligned}
\mathrm{Cov}\left(\mathbb{E}[\rho|s_{i,\rho}], \mathbb{E}[\rho|s_{j,\rho}]|\bar{\tau}_{s,\rho}\right) &= \mathrm{Cov}\left(\bar{\tau}_{s,\rho,i}s_{i,\rho} + (1 - \bar{\tau}_{s,\rho,i})\mu_\rho, \bar{\tau}_{s,\rho,j}s_{j,\rho} + (1 - \bar{\tau}_{s,\rho,j})\mu_\rho\right) \\
&= \mathrm{Cov}\left(\bar{\tau}_{s,\rho,i}s_{i,\rho}, \bar{\tau}_{s,\rho,j}s_{j,\rho}\right) \\
&= \bar{\tau}_{s,\rho,i}\bar{\tau}_{s,\rho,j}\mathrm{Var}(\rho) \\
&= \frac{\bar{\tau}_{s,\rho,i}\bar{\tau}_{s,\rho,j}}{\tau_\rho} = \frac{\bar{\tau}^2_{s,\rho}}{\tau_\rho} \text{ (when symmetric)}
\end{aligned}$$

□

Proof of expectation (19).

$$\begin{aligned}
\mathbb{E}[p_{i,2}|\bar{\tau}_{s,i}] &= \mathbb{E}\left[\frac{1}{2b}\left(a + bc_i + e\mathbb{E}[p_{j,2}|\rho, p_{i,1}, p_{j,1}]\right)\middle|\, \bar{\tau}_{s,i}\right] \\
&= \frac{a}{2b} + \frac{1}{2}(\mu_\rho + \mu_\theta) + \frac{e}{2b}\left(\frac{1}{4b^2 - e^2}\left((2b+e)a + 2b^2\mathbb{E}\left[\mathbb{E}[c_j|\rho, p_{j,1}]\right|\bar{\tau}_{s,i}] + be\mathbb{E}\left[\mathbb{E}[c_j|\rho, p_{j,1}]\right|\bar{\tau}_{s,i}]\right)\right) \\
&= \frac{a}{2b} + \frac{1}{2}(\mu_\rho + \mu_\theta) + \frac{e}{2b}\left(\frac{1}{4b^2 - e^2}\left((2b+e)a + (2b^2 + be)(\mu_\rho + \mu_\theta)\right)\right) \\
&= \frac{a}{2b} + \frac{1}{2}(\mu_\rho + \mu_\theta) + \frac{e}{2b}\left(\frac{a + b(\mu_\rho + \mu_\theta)}{2b - e}\right) \\
&= \frac{a + b(\mu_\rho + \mu_\theta)}{2b - e}
\end{aligned}$$

□

Proof of expectation (20).

$$\begin{aligned}\text{Var}(p_{i,2}|\bar{\tau}) &= \text{Var}\left(\frac{1}{2b}\left(a + bc_i + e\mathbb{E}[p_{j,2}|\rho, p_{i,1}, p_{j,1}]\right)\bigg|\,\bar{\tau}\right) \\ &= \frac{1}{4}\text{Var}(c_i) + \frac{e^2}{4b^2}\text{Var}\left(\mathbb{E}[p_{j,2}|\rho, p_{i,1}, p_{j,1}]|\bar{\tau}\right) - \frac{e}{4b}\text{Cov}\left(c_i, \mathbb{E}[p_{j,1}|\rho, p_{i,1}, p_{j,1}]|\bar{\tau}\right)\end{aligned}$$

□

Proof of expectation (21).

$$\begin{aligned}\text{Cov}(p_{i,2}, p_{j,2}|\bar{\tau}) &= \text{Cov}\left(\frac{1}{2b}\left(a + bc_i + e\mathbb{E}[p_{j,2}|\rho, p_{i,1}, p_{j,1}]\right), \frac{1}{2b}\left(a + bc_j + e\mathbb{E}[p_{i,2}|\rho, p_{i,1}, p_{j,1}]\right)\bigg|\,\bar{\tau}\right) \\ &= \frac{1}{4}\text{Cov}(c_i, c_j|\bar{\tau}) + \frac{e}{2b}\text{Cov}\left(c_i, \mathbb{E}[p_{j,2}|\rho, p_{i,1}, p_{j,1}]|\bar{\tau}\right) \\ &+ \frac{e^2}{4b^2}\text{Cov}\left(\mathbb{E}[p_{i,2}|\rho, p_{i,1}, p_{j,1}], \mathbb{E}[p_{j,2}|\rho, p_{i,1}, p_{j,1}]|\bar{\tau}\right)\end{aligned}$$

□

Proof of expectation (22).

$$\begin{aligned}\text{Var}\left(\mathbb{E}[p_{j,2}|\rho, p_{i,1}, p_{j,1}]|\bar{\tau}\right) &= \text{Var}\left(\frac{1}{4b^2 - e^2}\left((2b + e)a + 2b^2\mathbb{E}[c_j|\rho, p_{j,1}] + be\mathbb{E}[c_i|\rho, p_{i,1}]\right)\right) \\ &= \left(\frac{1}{4b^2 - e^2}\right)^2 \text{Var}\left(2b^2\mathbb{E}[c_j|\rho, p_{j,1}] + be\mathbb{E}[c_i|\rho, p_{i,1}]|\bar{\tau}\right) \\ &= \frac{1}{(4b^2 - e^2)^2}\left\{4b^4\text{Var}\left(\mathbb{E}[c_j|\rho, p_{j,1}]|\bar{\tau}\right) + b^2e^2\text{Var}\left(\mathbb{E}[c_i|\rho, p_{i,1}]|\bar{\tau}\right)\right. \\ &\left. -2(2b^2)(be)\text{Cov}\left(\mathbb{E}[c_j|\rho, p_{j,1}], \mathbb{E}[c_i|\rho, p_{i,1}]|\bar{\tau}\right)\right\}\end{aligned}$$

□

Proof of expectation (23).

$$\begin{aligned}\text{Cov}\left(\mathbb{E}[p_{i,2}|\rho, p_1], \mathbb{E}[p_{j,2}|\rho, p_1]|\bar{\tau}\right) &= \frac{1}{(4b^2 - e^2)^2}\left[4b^4\text{Cov}\left(\mathbb{E}[c_j|\rho, p_{j,i}], \mathbb{E}[c_i|\rho, p_{i,1}]\right) + 2b^3e\text{Var}\left(\mathbb{E}[c_j|\rho, p_{i,1}]|\bar{\tau}\right)\right. \\ &\left. +b^2e^2\text{Cov}\left(\mathbb{E}[c_j|\rho, p_{j,i}], \mathbb{E}[c_i|\rho, p_{i,1}]\right) + 2b^3e\text{Var}\left(\mathbb{E}[c_i|\rho, p_{i,1}]|\bar{\tau}\right)\right] \\ &= \frac{4b^4 + b^2e^2}{(4b^2 - e^2)^2}\text{Cov}\left(\mathbb{E}[c_j|\rho, p_{j,i}], \mathbb{E}[c_i|\rho, p_{i,1}]|\bar{\tau}\right) \\ &+ \frac{4b^3e}{(4b^2 - e^2)^2}\text{Var}\left(\mathbb{E}[c_i|\rho, p_{i,1}]|\bar{\tau}\right)\end{aligned}$$

□

Proof of expectation (24).

$$\begin{aligned}
\operatorname{Var}\left(\mathbb{E}[c_k|\rho, p_{k,1}]|\bar{\tau}\right) &= \operatorname{Var}\left(\mu_\rho + \mu_\theta + (1-\kappa_k\bar{\tau}_{s,\rho,k}p_\rho(\bar{\tau}))(\rho-\mu_\rho)\right. \\
&+ \left.(p_{k,1} - (p_0(\bar{\tau}) + p_\rho(\bar{\tau})\mu_\rho + p_\theta(\bar{\tau})\mu_\theta))\kappa_k \right| \bar{\tau}) \\
&= \kappa^2\operatorname{Var}(p_{k,1}|\bar{\tau}) + (1-\kappa\bar{\tau}_{s,\rho,k}p_\rho(\bar{\tau}))^2\operatorname{Var}(\rho|\bar{\tau}) \\
&- 2(1-\kappa\bar{\tau}_{s,\rho,k}p_\rho(\bar{\tau}))\kappa\operatorname{Cov}(\rho, p_{k,1}|\bar{\tau}) \\
&= \kappa^2 p_\theta^2(\bar{\tau})\operatorname{Var}(\mathbb{E}[\theta_i|s_{i,\theta}]|\bar{\tau}) + \kappa^2 p_\rho^2(\bar{\tau})\bar{\tau}_{s,\rho}^2(\operatorname{Var}(\rho|\bar{\tau}) + \operatorname{Var}(\varepsilon_{k,\rho}|\bar{\tau})) \\
&+ (1-\kappa\bar{\tau}_{s,\rho,k}p_\rho(\bar{\tau}))^2\operatorname{Var}(\rho|\bar{\tau}) - 2(1-\kappa\bar{\tau}_{s,\rho,k}p_\rho(\bar{\tau}))\kappa p_\rho(\bar{\tau})\bar{\tau}_{s,\rho,k}\operatorname{Var}(\rho|\bar{\tau}) \\
&= \operatorname{Var}(\rho|\bar{\tau}) + \kappa^2\bar{\tau}_{s,\rho,i}^2 p_\rho^2(\bar{\tau})\operatorname{Var}(\varepsilon_{i,\rho}|\bar{\tau}) + \kappa^2 p_\theta^2(\bar{\tau})\operatorname{Var}\left(\mathbb{E}[\theta_i|s_{i,\theta}]|\bar{\tau}\right)
\end{aligned}$$

□

Proof of expectation (25).

$$\begin{aligned}
\operatorname{Cov}\left(\mathbb{E}[c_j|\rho, p_{j,i}], \mathbb{E}[c_i|\rho, p_{i,1}]\right) &= \operatorname{Cov}((1-\kappa_j\bar{\tau}_{s,\rho,j}p_\rho(\bar{\tau}))\rho + p_{j,1}\kappa_j, (1-\kappa_i\bar{\tau}_{s,\rho,i}p_\rho(\bar{\tau}))\rho + p_{i,1}\kappa_i) \\
&= (1-\kappa_j\bar{\tau}_{s,\rho,j}p_\rho(\bar{\tau}))(1-\kappa_i\bar{\tau}_{s,\rho,i}p_\rho(\bar{\tau}))\operatorname{Var}(\rho|\bar{\tau}) \\
&+ \kappa_i(1-\kappa_j\bar{\tau}_{s,\rho,j}p_\rho(\bar{\tau}))\operatorname{Cov}(\rho, p_{i,1}|\bar{\tau}) + \kappa_j(1-\kappa_i\bar{\tau}_{s,\rho,i}p_\rho(\bar{\tau}))\operatorname{Cov}(\rho, p_{j,1}|\bar{\tau}) \\
&+ \kappa^2\operatorname{Cov}(p_{i,1}, p_{j,1}|\bar{\tau}) \\
&= (1-\kappa\bar{\tau}_{s,\rho,i}p_\rho(\bar{\tau}))^2\operatorname{Var}(\rho|\bar{\tau}) + 2\kappa(1-\kappa\bar{\tau}_{s,\rho,i}p_\rho(\bar{\tau}))\operatorname{Cov}(\rho, p_{i,1}|\bar{\tau}) \\
&+ \kappa^2\operatorname{Cov}(p_{i,1}, p_{j,1}|\bar{\tau}) \text{ (when symmetric)} \\
&= (1-\kappa\bar{\tau}_{s,\rho,i}p_\rho(\bar{\tau}))^2\operatorname{Var}(\rho|\bar{\tau}) + 2(1-\kappa\bar{\tau}_{s,\rho,i}p_\rho(\bar{\tau}))\kappa\bar{\tau}_{s,\rho,i}p_\rho(\bar{\tau})\operatorname{Var}(\rho|\bar{\tau}) \\
&+ \kappa^2\bar{\tau}_{s,\rho,i}^2 p_\rho^2(\bar{\tau})\operatorname{Var}(\rho|\bar{\tau}) \\
&= (1-\kappa\bar{\tau}_{s,\rho,i}p_\rho(\bar{\tau}) + \kappa\bar{\tau}_{s,\rho,i}p_\rho(\bar{\tau}))^2\operatorname{Var}(\rho|\bar{\tau}) \\
&= \operatorname{Var}(\rho|\bar{\tau})
\end{aligned}$$

□

Proof of expectation (26).

$$\begin{aligned}
\operatorname{Cov}(\rho, p_{i,1}|\bar{\tau}) &= \operatorname{Cov}\left(\rho, p_0(\bar{\tau}) + p_\rho(\bar{\tau})\mathbb{E}[\rho|s_{i,\rho}] + p_\theta(\bar{\tau})\mathbb{E}[\theta_i|s_{i,\theta}]|\bar{\tau}\right) \\
&= p_\rho(\bar{\tau})\operatorname{Cov}\left(\rho, \mathbb{E}[\rho|s_{i,\rho}]|\bar{\tau}\right) \\
&= p_\rho(\bar{\tau})\bar{\tau}_{s,\rho,i}\operatorname{Var}(\rho|\bar{\tau})
\end{aligned}$$

□

Proof of expectation (27).

$$\begin{aligned}\mathrm{Cov}\left(c_i, \mathbb{E}[p_{j,2}|\rho, p_{i,1}, p_{j,1}]|\bar{\tau}\right) &= \mathrm{Cov}\left(c_i, \frac{1}{4b^2-e^2}\left((2b+e)a + 2b^2\mathbb{E}[c_j|\rho, p_{j,1}] + be\mathbb{E}[c_j|\rho, p_{i,1}]\right)\right) \\ &= \frac{2b^2}{4b^2-e^2}\mathrm{Cov}\left(c_i, \mathbb{E}[c_j|\rho, p_{j,1}]|\bar{\tau}\right) + \frac{be}{4b^2-e^2}\mathrm{Cov}\left(c_i, \mathbb{E}[c_i|\rho, p_{i,1}]|\bar{\tau}\right)\end{aligned}$$

□

Proof of expectation (28).

$$\begin{aligned}\mathrm{Cov}\left(c_i, \mathbb{E}[c_j|\rho, p_{j,1}]|\bar{\tau}\right) &= \mathrm{Cov}\left(\rho, (1-\kappa\bar{\tau}_{s,\rho,i}p_\rho(\bar{\tau}))\rho + p_\rho(\bar{\tau})\mathbb{E}[\rho|s_{j,\rho}]|\bar{\tau}\right) \\ &= (1-\kappa\bar{\tau}_{s,\rho,i}p_\rho(\bar{\tau}))\mathrm{Var}(\rho|\bar{\tau}) + \kappa\bar{\tau}_{s,\rho,i}p_\rho(\bar{\tau})\mathrm{Var}(\rho|\bar{\tau}) \\ &= \mathrm{Var}(\rho|\bar{\tau})\end{aligned}$$

□

Proof of expectation (29).

$$\begin{aligned}\mathrm{Cov}\left(c_i, \mathbb{E}[c_i|\rho, p_{i,1}]|\bar{\tau}\right) &= \mathrm{Cov}\left(\rho+\theta_i, (1-\kappa\bar{\tau}_{s,\rho,i}p_\rho(\bar{\tau}))\rho + p_\rho(\bar{\tau})\mathbb{E}[\rho|s_{j,\rho}] + p_\theta(\bar{\tau})\mathbb{E}[\theta_i|s_{i,\theta}]|\bar{\tau}\right) \\ &= \mathrm{Var}(\rho|\bar{\tau}) + \kappa\bar{\tau}_{s,\theta,i}p_\theta(\bar{\tau})\mathrm{Var}(\theta_i|\bar{\tau})\end{aligned}$$

□

Proof of expectation (30).

$$\begin{aligned}\mathrm{Var}\left(\mathbb{E}[p^c_{j,2}|\rho, p_{i,1}, p_{j,1}]|\bar{\tau}\right) &= \frac{1}{(4b^2-e^2)^2}\left\{4b^4\mathrm{Var}\left(\mathbb{E}[c_j|\rho, p^c_{j,1}]|\bar{\tau}\right) + b^2e^2\mathrm{Var}\left(\mathbb{E}[c_i|\rho, p^c_{i,1}]|\bar{\tau}\right)\right. \\ &\left.-2(2b^2)(be)\mathrm{Cov}\left(\mathbb{E}[c_j|\rho, p^c_{j,1}], \mathbb{E}[c_i|\rho, p^c_{i,1}]|\bar{\tau}\right)\right\} \\ &= \frac{1}{(4b^2-e^2)^2}\left\{4b^4\mathrm{Var}\left(\mathbb{E}[c_j|\rho, s_{j,\theta}]|\bar{\tau}\right) + b^2e^2\mathrm{Var}\left(\mathbb{E}[c_i|\rho, s_{i,\theta}]|\bar{\tau}\right)\right. \\ &\left.-2(2b^2)(be)\mathrm{Cov}\left(\mathbb{E}[c_j|\rho, s_{j,\theta}], \mathbb{E}[c_i|\rho, s_{i,\theta}]|\bar{\tau}\right)\right\} \\ &= \frac{1}{(4b^2-e^2)^2}\left\{4b^4\left(\mathrm{Var}(\rho|\bar{\tau}) + \mathrm{Var}\left(\mathbb{E}[\theta_j|s_{j,\theta}]|\bar{\tau}\right)\right)\right. \\ &\left.+b^2e^2\left(\mathrm{Var}(\rho|\bar{\tau}) + \mathrm{Var}\left(\mathbb{E}[\theta_i|s_{i,\theta}]|\bar{\tau}\right)\right) - 4b^3e\mathrm{Var}(\rho|\bar{\tau})\right\}\end{aligned}$$

□

Proof of expectation (31).

$$\begin{aligned}
\text{Cov}\left(c_i, \mathbb{E}[p^c_{j,2}|\rho, p^c_{i,1}, p^c_{j,1}]|\bar{\tau}\right) &= \frac{2b^2}{4b^2-e^2}\text{Cov}\left(c_i, \mathbb{E}[c_j|\rho, p^c_{j,1}]|\bar{\tau}\right) + \frac{be}{4b^2-e^2}\text{Cov}\left(c_i, \mathbb{E}[c_i|\rho, p^c_{i,1}]|\bar{\tau}\right) \\
&= \frac{2b^2}{4b^2-e^2}\text{Cov}\left(c_i, \mathbb{E}[c_j|\rho, s_{j,\theta}]|\bar{\tau}\right) + \frac{be}{4b^2-e^2}\text{Cov}\left(c_i, \mathbb{E}[c_i|\rho, s_{i,\theta}]|\bar{\tau}\right) \\
&= \frac{2b^2}{4b^2-e^2}\text{Var}\left(\rho|\bar{\tau}\right) + \frac{be}{4b^2-e^2}\left(\text{Var}(\rho|\bar{\tau}) + \text{Cov}\left(\theta_i, \mathbb{E}[\theta_i|s_{i,\theta}]|\bar{\tau}\right)\right) \\
&= \frac{1}{4b^2-e^2}\left(2b^2\text{Var}(\rho|\bar{\tau}) + be\left(\text{Var}(\rho|\bar{\tau}) + \bar{\tau}_{s,\theta,i}\text{Var}(\theta_i|\bar{\tau})\right)\right)
\end{aligned}$$

□

Proof of expectation (32).

$$\begin{aligned}
\text{Cov}\left(\mathbb{E}[p^c_{i,2}|\cdot], \mathbb{E}[p^c_{j,2}|\cdot]|\bar{\tau}\right) = \frac{4b^2e^2+e^4}{4(4b^2-e^2)^2}&\text{Cov}\left(\mathbb{E}[c_j|\rho, p^c_{j,i}], \mathbb{E}[c_i|\rho, p^c_{i,1}]|\bar{\tau}\right) \\
+ \frac{be^3}{(4b^2-e^2)^2}&\text{Var}\left(\mathbb{E}[c_i|\rho, p^c_{i,1}]|\bar{\tau}\right)
\end{aligned}$$

□

C Bounds on values

The following inequalities are used throughout the paper.

$$\beta \in \left[0, \frac{1}{3}\right] \tag{33}$$

$$\frac{b-e}{2b-e} \in \left[0, \frac{2}{3}\right] \tag{34}$$

$$\left(\frac{b-e}{2b-e}\right)\beta \in \left[0, \frac{2}{9}\right] \tag{35}$$

$$p_\theta \in \left[\frac{1}{3}, \frac{1}{2}\right] \tag{36}$$

$$\kappa \in [0, 3] \tag{37}$$

$$\left|\frac{be}{4b^2-e^2}\right|\kappa \in [0, 1] \tag{38}$$

$$\beta\kappa \in \left[0, \left|\frac{e}{b}\right|\right] \subseteq [0, 1] \tag{39}$$

$$p_\rho \in \left[\frac{1}{9}, \frac{1}{2}\right] \text{ (when } e < 0) \tag{40}$$

$$p_\rho \in [0.46, 1] \text{ (when } e > 0) \tag{41}$$

C.1 Proofs of bounds

Proof of inequality (33). Since $|e| \leq b$, $\beta = e^2/(4b^2 - e^2) \geq 0$. To establish the upper bound, note that the numerator is increasing in e^2 and the denominator is decreasing in e^2, so the maximum value of β will be attained when e^2 is at its maximum. Since $e^2 \leq b^2$, it follows that $\beta \leq 3$. □

Proof of inequality (34). Since $|e| \leq b$, it is immediate that $(b-e)/(2b-e) \geq 0$. To establish the upper bound we examine the first derivative of the expression,[19]

$$\frac{-(2b-e)+(b-e)}{(2b-e)^2} = -\frac{b}{(2b-e)^2} < 0.$$

Then the derivative is negative everywhere, and the expression is maximized when e is at its minimum, $e = -b$. This gives

$$\frac{b-(-b)}{2b-(-b)} = \frac{2}{3}.$$

□

[19] Basic intuition about fractions is sufficient for this maximization. We find that straightforward calculus is simpler to analyze.

Proof of inequality (35). This follows directly from inequalities (33) and (34). □

Proof of inequalities (36) *and* (37). Since $\beta\kappa \geq 0$ and $p_\theta = 1/(2+\beta\kappa)$, it must be that $p_\theta \leq 1/2$. Further, p_θ will be minimized when $\beta\kappa$ is maximized. Looking at κ in isolation,

$$\kappa = \frac{\sigma_\theta^2 \bar{\tau}_{s,\theta} p_\theta}{\sigma_{s,\rho}^2 \bar{\tau}_{s,\rho}^2 p_\rho^2 + \left(\sigma_\theta^2 + \sigma_{s,\theta}^2\right) \bar{\tau}_{s,\theta}^2 p_\theta^2}.$$

All involved terms are positive, so κ can be bounded above by assuming that $\bar{\tau}_{s,\rho} = 0$. This gives

$$\kappa \leq \frac{\sigma_\theta^2 \bar{\tau}_{s,\theta} p_\theta}{\left(\sigma_\theta^2 + \sigma_{s,\theta}^2\right) \bar{\tau}_{s,\theta}^2 p_\theta^2} = \frac{\sigma_\theta^2 \bar{\tau}_{s,\theta} p_\theta}{\sigma_\theta^2 \bar{\tau}_{s,\theta} p_\theta^2} = \frac{1}{p_\theta}.$$

Let $\underline{p}_\theta$ be the minimum feasible value of p_θ and $\overline{\beta} = 1/3$ be the maximum feasible value of β; then $\kappa \leq 1/\underline{p}_\theta$. It follows that

$$p_\theta \geq \frac{1}{2 + \frac{\beta}{\underline{p}_\theta}} \quad \Longrightarrow \quad \underline{p}_\theta \geq \frac{1}{2 + \frac{\overline{\beta}}{\underline{p}_\theta}}.$$

This gives

$$2\underline{p}_\theta + \overline{\beta} \geq 1 \quad \Longrightarrow \quad \underline{p}_\theta \geq \frac{1}{3}.$$

Then $p_\theta \geq 1/3$. It follows that $\kappa \leq 3$. Since $|e| \leq b$, $be/(4b^2 - e^2) \leq 1/3$, hence

$$\left(\frac{be}{4b^2 - e^2}\right)\kappa \leq \left(\frac{1}{3}\right) 3 = 1.$$

□

Proof of inequality (38). Note that

$$\left|\frac{be}{4b^2 - e^2}\right| = \frac{b\,|e|}{4b^2 - |e|^2}.$$

The numerator is increasing in $|e|$ while the denominator is decreasing in $|e|$, thus this ratio will be maximized when $|e|$ is maximized, or when $|e| = b$. Then

$$\left|\frac{be}{4b^2 - e^2}\right| \leq \frac{b^2}{4b^2 - b^2} = \frac{1}{3}.$$

Since inequality (37) gives $\kappa \leq 3$, inequality (38) follows. □

Proof of inequality (39). Note that

$$\beta = \frac{e}{b}\left(\frac{be}{4b^2 - e^2}\right).$$

Then inequality (39) follows immediately from inequality (38). □

Proof of inequalities (40) *and* (41). Recall the equilibrium equation for p_ρ,

$$p_\rho = \frac{1 - \left(\frac{1-r}{2-r}\right)\beta\kappa}{(2 - r\bar{\tau}_{s,\rho}) - \frac{1}{2}\left(1 - \bar{\tau}_{s,\rho}\right)\beta^2\kappa^2}, \text{ where } r = \frac{e}{b}.$$

By inequality (39), $\beta\kappa \leq |r|$, so the bound on the denominator will depend on the sign of r.

When $r < 0$, the denominator is bounded below by $2 - \beta^2\kappa^2/2$ and above by $2 - r$. The numerator is bounded above by $1 - \beta\kappa/2$. This gives

$$\begin{aligned} p_\rho &\leq \frac{1 - \frac{1}{2}\beta\kappa}{2 - \frac{1}{2}\beta^2\kappa^2} \\ &= \frac{2 - \beta\kappa}{4 - \beta^2\kappa^2} \\ &= \frac{1}{2 + \beta\kappa} \leq \frac{1}{2}; \end{aligned} \qquad \begin{aligned} p_\rho &\geq \frac{1 - \left(\frac{1-r}{2-r}\right)\beta\kappa}{2 - r} \\ &\geq \frac{(2-r) - (1-r)}{(2-r)^2} \\ &= \frac{1}{(2-r)^2} \geq \frac{1}{9}. \end{aligned}$$

When $r > 0$, the denominator is bounded below by $2 - r$ and above by $2 - \beta^2\kappa^2/2$. The numerator is bounded below by $1 - \beta\kappa/2$. This gives

$$\begin{aligned} p_\rho &\leq \frac{1 - \left(\frac{1-r}{2-r}\right)\beta\kappa}{2 - r} \\ &\leq \frac{(2-r) - (1-r)\,r}{(2-r)^2} \\ &= \frac{2 - 2r + r^2}{(2-r)^2} \leq 1; \end{aligned} \qquad \begin{aligned} p_\rho &\geq \frac{1 - \left(\frac{1-r}{2-r}\right)\left(\frac{r^2}{4-r^2}\right)\kappa}{2 - \frac{1}{2}\left(\frac{r^2}{4-r^2}\right)^2\kappa^2} \\ &\geq \frac{1 - \left(\frac{1-r}{2-r}\right)\left(\frac{r^2}{4-r^2}\right)\left(\frac{2}{1 - r^2/(4-r^2)}\right)}{2} \\ &\geq \frac{1}{2} - \left(\frac{1-r}{2-r}\right)\left(\frac{r^2}{4-2r^2}\right) \geq\approx 0.46. \end{aligned}$$

□