Repeated Contests with Private Information*

Greg Kubitz†

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Abstract

In contests with private information, low ability contestants prefer to appear strong while high ability contestants prefer to appear weak. In a repeated contest, this leads to an equilibrium with partial pooling in the initial stage, i.e. the output of a contestant is not strictly monotonic in ability. Higher stakes in a future contest increases pooling and the likelihood that a low ability contestant wins against an opponent of higher ability. Increasing the stakes of the initial contest decreases pooling, reducing the expected competitiveness of a future contest. For a class of cost of effort functions and a fixed prize pool, unequal prizes over the two contests can increase total expected output while decreasing the expected payoffs of contestants relative to contests with equal prizes.

1 Introduction

Contests are frequently used to stimulate effort from economic agents. These contests are often dynamic and offer multiple prizes, as in the case of repeated employee competitions and multi-stage tournaments. In both settings, there is extensive literature discussing how to best design contests. However, the behavior of contestants in dynamic contests is not fully characterized when the contestants have private information about productive ability or the value of winning the tournament.[1]

In this paper, we study repeated contests in a framework designed to capture both moral hazard (hidden effort choice) and adverse selection (privately known abilities). That is, when contestants' abilities are private information, the contestants must consider the signaling effect that exerting effort in early contests will have in future contests. Contrary to the conventional wisdom that all contestants want to appear strong to their opponents, high ability contestants instead prefer to appear weak. The desire to both win the current contest and be in a preferred position for future contests create countervailing incentives leading to strategies that are not strictly monotone in ability. When

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†Queensland University of Technology, gregory.kubitz@qut.edu.au

[1] A model of repeated all-pay auctions where bidders have private valuation is contained within the current model.

future contests have relatively higher stakes, additional pooling in the initial contest increases the likelihood of a low ability contestant beating an opponent with higher ability. On the other hand, higher initial stakes reduces pooling, separating contestants by ability and leading to less competitive future contests.

We consider the simplest setting that captures the signaling incentives of repeated contests: two contestants, who have either low or high ability, competing in two successive contests. In each contest, the contestants exert effort with the goal of producing the most output. The player who does so wins a prize. The amount of effort it takes to produce output depends on individual ability, which is privately known by each contestant. After the first contest, the output of each contestant is publicly observed and players can update their beliefs about their opponents' ability. Given this additional information, contestants choose a new level of effort for the second contest. The contestants choose their effort levels to maximize their total payoff over the two contests.

We show there is a unique symmetric equilibrium for this repeated contest game. The equilibrium strategies of both high and low ability contestants reflect the trade-off between success in the first contest and optimal positioning for success in the second contest. The complementarity of ability and effort would lead to a high level of output from high ability contestants and a low level of output from low ability contestants if there was only a single contest. However, entering the second contest, a contestant with high ability will always prefer to have his opponent believe they have low ability. Likewise, a contestant with low ability wants to appear to have high ability. Concern about the outcomes in both contests leads to an equilibrium that has partial pooling in the first contest, i.e. there is a range of outputs which can be produced by either low ability or high ability contestants. Low ability players who produce output in this range are *bluffing* while high ability players who do so are *sandbagging*.[2]

Both tactics are used in the first contest with the purpose of depressing the effort of opponents in the second contest. In particular, contestants care primarily about the effort choice of opponents who have similar ability. For example, a low ability opponent will be discouraged when they observe high output in the first contest, leading them to put less effort into the second contest. Therefore, low ability contestants have an incentive to put a high level of effort into the first contest. On the other hand, a high ability opponent would increase their effort in the second contest if they observe a high output in the first contest. In order to avoid this escalation, high ability contestants

[2]The terms *sandbag* and *bluff* are used in the literature to describe a player signaling to his opponent that he is weak when he is actually strong and strong when he is actually weak, respectively. These terms originate from the game of poker. In poker, *sandbagging* is when a player calls or does not increase the pot when he believes he has the better hand. *Bluffing* is when a player bids up the pot when he does not think he has the best hand.

will exert relatively little effort in the first contest.

Given a homogeneous cost function of effort, the unique equilibrium can be constructed. We use the construction to determine the welfare impact of the prize allocation over the two contests. In particular, the prizes of the two contests impact the relative strength of the countervailing incentives. A large prize in the first contest increases the incentive for high ability contestants to separate from low ability contestants. This increases the probability that one contestant will enter the second contest in a stronger position than the other, reducing the competitiveness and expected output of this contest. On the other hand, a large prize in the second contest increases the incentive to pool in the first contest and the probability of a victory by a low ability contestant. Additionally, while bluffing in the first contest increases expected output of low ability contestants, sandbagging by high ability contestants has the opposite effect.

Lastly, we consider how the expected output and expected payoffs of the contestants are impacted by the allocation of fixed prize purse over two contests. We show with numerical examples that a similar prize in each contest will tend to decrease expected output and increase expected payoffs of contestants relative to an asymmetric allocation of prizes. Prize asymmetry lessens the impact of reduced competition in the second contest either directly by lowering the stakes of the contest or indirectly by encouraging pooling in the first contest with a small first contest prize. On the other hand, contestants benefit from both reduced competition in the second contest and complementary strategic effects in the first contest: when low ability contestants have a strong incentive to bluff, it makes it easier for high ability contestants to sandbag.

Uniqueness of equilibrium in dynamic games with signaling is not common and stems from the countervailing incentives in the first contest. To derive this equilibrium, we first use an adaptation of the construction in Siegel (2014) to find the unique equilibrium of the second contest subgame for any set of abilities and beliefs that emerge from the first contest. The incentives to sandbag and bluff clearly emerge from the equilibrium payoffs in the second contest. For high ability contestants, expected payoffs strictly decrease in the other contestant's belief about their ability. For low ability contestants, these payoffs strictly increase in the opponent's belief. The desire of each type of contestant to appear as the other type not only leads to a unique partial pooling equilibrium in the first contest, it also rules out any additional equilibria. In signaling games, undesirable off-equilibrium path beliefs can often be used to construct additional perfect Bayesian equilibria. In this game, however, there are no beliefs that are undesirable to both high and low ability contestants. Therefore any belief used in an off-path output of the first contest will cause a deviation that unravels the potential equilibrium.

This paper contributes to an extensive literature investigating signal jamming in dynamic competitions. In a majority of these games, competitors have an incentive to bluff in order to discourage opponents. This behavior is observed in labor competitions as over working before a midterm evaluation (Ederer (2010)), in dynamic auctions as jump bidding (Avery (1998)) and in duopoly competition as excess production when firm costs are uncertain (Mirman et al. (1993) and Bonatti et al. (2017)).[3] Sandbagging, as described in Rosen (1986), is often used to lull opponents into a false sense of security. This type of signal-jamming also appears as bid-shading in repeated first price auctions, see Ortega Reichert (2000) and Bergemann and Hörner (2018).

In our setting, the incentives for strong competitors to sandbag and weak competitors to bluff is due to (i) the existence of private information, (ii) the partial revelation of this information through actions during the competition and (iii) the all-pay nature of the contest. This two directional distortion of strategies is also identified in Hörner and Sahuguet (2007) where bidders are able to signal their value through a fixed jump bid prior to an all-pay auction.[4] Bidders with moderate values will sometimes use this jump bid while bidders with high values may not. Similarly, in a multi-stage all-pay auction with elimination, Zhang and Wang (2009) shows non-existence of a separating equilibrium when winners bids are revealed prior to the following stage. Our paper clarifies how the incentives that arise in these settings impact the strategies of contestants within the competition itself, both before and after information is revealed.[5]

Information design in contests considers the optimal disclosure policy of the contest designer. This information may include private information about contestant types prior to a contest, as in Zhang and Zhou (2016), Lu et al. (2018) and Serena (2017), or early stage outcome information released during a multistage contest, as in Zhang and Wang (2009), Ederer (2010), Klein and Schmutzler (2017) and Aoyagi (2010). The model presented in this paper allows for quantitative analysis of how release of output information impacts strategies in a dynamic contest when abilities are private information.

Information about types or outcomes in contests often create asymmetries that reduce subsequent effort. The reduction of output from asymmetric contestant types is

[3]In complete information games with sequential moves, increased effort has a similar impact on opposing contestants that follow, see Hinnosaar (2018).

[4]Additional work that allow for costly signaling prior to a competition stage include Denter et al. (2019) and Heijnen and Schoonbeek (2017). Both Kovenock et al. (2015) and Wu and Zheng (2017) examine contestants willingness to share verifiable information before a contest. Signaling through coalition formation prior to a competition is studied in Konrad and Morath (2018).

[5]Repeated contests with private information are also studied in Münster (2009), where contestants may or may not value winning the contest. This leads to an equilibrium where contestants who do value the contest will sometimes not participate in the first contest but the incentive to bluff is not captured.

well known, see for example Baye et al. (1993), Che and Gale (2003), Terwiesch and Xu (2008) and Siegel (2010). It differs from the discouragement effect caused by falling behind in early stages of a contest of complete information and uncertain outcomes as in Harris and Vickers (1987) and Konrad and Kovenock (2009).[6] A contest designer can alleviate the reduction in effort from either type of discouragement by committing to bias later stages of a dynamic contest as in Ridlon and Shin (2013) and Barbieri and Serena (2018). In the current paper, the relative size of the prizes in each contest impacts the average asymmetry in the second contest but does not favor one contestant over the other.

The rest of the paper is organized as follows. In section 2 we formalize the repeated contest model. In sections 3 and 4 we characterize the equilibrium of this model by backwards induction, focusing on the second contest in section 3 and the first contest in section 4. In section 5 we discuss welfare implications of the equilibrium and section 6 concludes. Proofs of all results are in appendix A.

2 Model

Two ex-ante identical contestants are independently endowed with ability, a^i, for $i = 1, 2$. Each contestant is equally likely to have low ability, $a = a_\ell$, or high ability, $a = a_h$. Ability is normalized so that $a_\ell = 1$ and $a_h > 1$, and the endowment of ability is private information for each contestant. After the initial draw of types, the abilities of the contestants are fully persistent.

There are two sequential contests in which the contestants compete by choosing effort, e, and producing output, x. Effort and ability are complimentary, and the output function takes the form $x(a, e) = ae$. At the end of the first contest, the outputs of each contestant from the first contest, (x_1^1, x_1^2), become public information. Contestants use this information to update their beliefs about their opponent's ability prior to competing in the second contest. This belief is denoted by $\mu^{-i}(x_1^i)$ and is the probability that contestant i has high ability given the information available to contestant $-i$ after the first contest. Because these outputs are commonly observed, contestants first order beliefs are sufficient for characterizing each contestant's information in the second contest.

In each contest, the contestant that produces the most output receives a prize, p. If the two contestants produce the same output, then the prize is given randomly, each contestant winning with equal probability. Both contestants bear the cost of their effort

[6] See Konrad (2012) for a survey of discouragement in dynamic contests under complete information.

in each contest regardless of the its outcome. This cost, $c(e)$, is the same for high and low ability contestants. The cost function is assumed to be twice differentiable on the non-negative reals, increasing and weakly convex, with the cost of zero effort being zero. The payoff of contestant i given the ability and effort choice of each contestant are

$$\tilde{\pi}^i(a^i, e^i, a^{-i}, e^{-i}) = \begin{cases} p - c(e^i), & x(a^{-i}, e^{-i}) < x(a^i, e^i) \\ p/2 - c(e^i), & x(a^{-i}, e^{-i}) = x(a^i, e^i) \\ -c(e^i), & x(a^{-i}, e^{-i}) > x(a^i, e^i) \end{cases}$$

Given an output of their opponent, the expected payoffs of each contestant is equal to the probability that the contestant wins less his cost of effort. Here we abuse notation and let $x^i = x(a^i, e^i)$ for $i = 1, 2$.

$$\mathbb{E}[\tilde{\pi}^i(a^i, e^i)] = p\left[\Pr(x^{-i} < x(a^i, e^i)) + \frac{1}{2}\Pr(x^{-i} = x(a^i, e^i))\right] - c(e^i).$$

Since contestants know their own ability and the relationship between effort and output is deterministic, choosing effort level is equivalent to choosing output.[7] Therefore, strategies are written in terms of output to ease comparisons of contestants with different abilities. Additionally, it puts contestants' strategies in terms of what their opponents will observe. With this in mind, we write contestants' strategies and payoffs in terms of output and describe the effort of players only in the context of providing intuition for the results. Expected payoffs for contestant i in a single contest are

$$\mathbb{E}[\pi^i(x^i, a^i)] = p\left[\Pr(x^{-i} < x^i) + \frac{1}{2}\Pr(x^{-i} = x^i)\right] - c(x^i/a^i), \text{ for } i = 1, 2.$$

In the repeated contest game, contestant i maximizes the sum of expected payoffs in the two contests by choosing output levels in each contest, x_1^i and x_2^i. For a given strategy of player $-i$, these payoffs are

$$\begin{aligned}\mathbb{E}[\pi^i(x_1^i, x_2^i, a^i)] &= \mathbb{E}[\pi_1^i(x_1^i, a^i)] + \mathbb{E}[\pi_2^i(x_2^i, a^i)|\mu^{-i}(x_1^i)] \\ &= p_1\left[\Pr(x_1^{-i} < x_1^i) + \frac{1}{2}\Pr(x_1^{-i} = x_1^i)\right] - c\left(x_1^i/a^i\right) \\ &+ p_2\left[\Pr(x_2^{-i} < x_2^i|\mu^{-i}(x_1^i)) + \frac{1}{2}\Pr(x_2^{-i} = x_2^i|\mu^{-i}(x_1^i))\right] - c(x_2^i/a^i).\end{aligned}$$

[7] Equivalent to the notion of private information about ability is private information about the cost of output. Additionally, if the cost of effort is linear, then this framework is equivalent to an all-pay auction where values are private information and bids are observed

3 Second Contest

After the first of two contests, each contestant will believe that their opponent has high ability with some probability. For each set of these probabilities, the equilibrium characterized in this section will be played in the second contest. Therefore, the expected payoffs of the contestants in this section will be equal to the continuation payoffs of the second contest in any perfect Bayesian equilibrium of the repeated contest game.

In this section, we name our two contestants the strong contestant and the weak contestant, so that $i = s, w$ and $\mu_s \geq \mu_w$, where $\mu_i = \Pr(a^i = a_h)$. Ex-ante, the strong contestant, is at least as likely to have high ability as the weaker contestant. However, this does not rule out the possibility of the weaker contestant having high ability or the stronger contestant having low ability, or both.

3.1 Strategies

The strategies of each contestant consist of output distributions for both low and high ability realizations. Due to the all pay nature of the contest, equilibrium strategies will be mixed. We define $L_i(x|\mu_i, \mu_{-i}) \equiv \Pr(x^i \leq x|a^i = a_\ell, \mu_i, \mu_{-i})$ and $H_i(x|\mu_i, \mu_{-i}) \equiv \Pr(x^i \leq x|a^i = a_h, \mu_i, \mu_{-i})$ which denote these respective distributions. The ex-ante output distribution of contestant i, defined as $F_i(x|\mu_i, \mu_{-i}) \equiv \Pr(x^i \leq x|\mu_i, \mu_{-i})$, also represents the ex-interim output distribution of player i from the perspective of player $-i$. Consistency of information sets requires $F_i(x|\mu_i, \mu_{-i}) = (1 - \mu_i)L_i(x|\mu_i, \mu_{-i}) + \mu_i H_i(x|\mu_i, \mu_{-i})$. For simplicity, we suppress the probabilities, (μ_i, μ_{-i}), from the notation of the output distributions.

Given an expected output distribution of their opponent, the best response set for contestant i with ability a^i is $BR_i(a^i) \equiv \{x : \mathbb{E}[\pi^i(x^i, a^i)] \geq \mathbb{E}[\pi^i(\tilde{x}^i, a^i)], \forall \tilde{x}^i \geq 0\}$. An equilibrium is a set of output distributions, $(L_s(x), H_s(x), L_w(x), H_w(x))$, that induce densities whose supports are subsets of the pertinent best response set.[8]

In any equilibrium, there can be no gaps in the best response sets and no outputs chosen with positive probability, except at zero. The expected output distributions of each contestant are continuous for positive outputs and the equilibrium is monotonic, i.e., best response sets are disjoint intervals for each ability level of a given contestant, with the set for high ability ranging over larger outputs than the set for low ability. The combined best response sets of each contestant must be the same interval for the strong and weak contestant. Since the strong player is more likely to be high ability,

[8]For example, the support of $h_s(x)$, the density induced from $H_s(x)$, must be a subset of $BR_s(a_h)$.

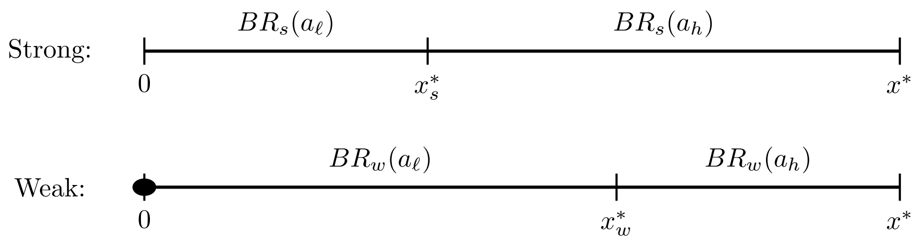


Figure 1: Representation of best response sets of the strong and weak players in the second contest.

the length of the best response set of high ability is longer for the strong player. The basic structure of these best response sets is shown in Figure 1.

Lemma 1. *In any equilibrium,* $BR_i(a_\ell) \cup BR_i(a_h) = [0, x^*]$ *for* $i = s, w$ *and* $x^* > 0$. *Output distributions,* $H_s(x), L_s(x), H_w(x)$, *and* $L_w(x)$, *are continuous on* $(0, x^*]$ *and the equilibrium is monotonic.*

While the fundamentals of the model are somewhat different to Siegel (2014), the general properties of the equilibrium strategies are the same.[9] In that all-pay auction setting, when values are independent there is a unique monotonic equilibrium. The monotonicity property holds in the current model as a player's marginal cost for a given output is ranked by type.

Proposition 1 (Unique Equilibrium - Second Contest)**.** *There is a unique equilibrium,* $(L_s^*(x), H_s^*(x), L_w^*(x), H_w^*(x))$, *where* $BR_i(a_\ell) = [0, x_i^*]$, $BR_i(a_h) = [x_i^*, x^*]$ *for* $i = s, w$ *and* $0 \leq x_s^* \leq x_w^* \leq x^*$.

3.2 Payoffs

For strategies in the first contest, the main objects of interest from the equilibrium of the second contest are the payoffs of the contestants. A contestant's second contest continuation value in the repeated contest game is a function of the contestant's ability and the probabilities of each contestant being high ability as viewed by their opponent. These payoffs are denoted as $v_i(\mu_i, \mu_{-i}, a^i) = \mathbb{E}[\pi^i(\hat{x}^i, a^i)]$ and are characterized below.

[9] A simple transformation connects our setting restricted to linear cost and that setting restricted to independent types.

Corollary 1. *The ex-interim expected payoff of each contestant is*

$$v_s(\mu_s, \mu_w, a_h) = v_w(\mu_w, \mu_s, a_h) = p_2(1 - \mu_w) - c\left(\frac{c^{-1}(p_2(1 - \mu_w))}{a_h}\right)$$
$$v_s(\mu_s, \mu_w, a_\ell) = p_2(\mu_s - \mu_w) - \left[c\left(\frac{c^{-1}(p_2(1 - \mu_w))}{a_h}\right) - c\left(\frac{c^{-1}(p_2(1 - \mu_s))}{a_h}\right)\right]$$
$$v_w(\mu_w, \mu_s, a_\ell) = 0.$$

For contestants who are high ability, the expected payoff is determined by the probability that the weaker contestant has high ability. Intuitively, high ability contestants are confident they can win, but increased competition, i.e. higher μ_w, increases how much effort they need to exert to do so. For contestants with low ability, expected payoffs are determined by how often the other contestant chooses no effort. While the stronger contestant will always exert positive effort, the weaker contestant exerts no effort with a probability that increases with the strength of the other contestant.

Contestants can affect their perceived strength in this second contest through their choice of output in the first contest. High ability contestants prefer to look weak entering the second contest as their expected payoffs decrease when the contest appears more competitive. On the other hand, the expected payoffs of low ability contestants increase when they appear strong to their opponent. These incentives, formalized in the proposition below, are a significant strategic force in the first contest and exist for any strategies in the first contest.

Proposition 2. *Let $F_{\mu_{-i}}(M) = \Pr(\mu_{-i} \leq M)$ be the belief distribution of player $-i$ resulting from the first contest and let $\underline{M} = \sup\{M | F_{\mu_{-i}}(M) = 0\}$ and $\overline{M} = \inf\{M | F_{\mu_{-i}}(M) = 1\}$. For all $\mu_i \in (\underline{M}, \overline{M})$, expected payoffs in the second contest decrease for high ability players as μ_i increases, $\frac{\partial}{\partial \mu_i}\mathbb{E}[v_i(\mu_i, \mu_{-i}, a_h)] < 0$, and increase with μ_i for low ability players, $\frac{\partial}{\partial \mu_i}\mathbb{E}[v_i(\mu_i, \mu_{-i}, a_\ell)] > 0$.*

The marginal effect of beliefs on expected payoffs in the second contest is given by

$$\frac{\partial}{\partial \mu_i}\mathbb{E}[v_i(\mu_i, \mu_{-i}, a_h)] = d(\mu_i)(F_{\mu_{-i}}(\mu_i) - 1) \text{ and } \frac{\partial}{\partial \mu_i}\mathbb{E}[v_i(\mu_i, \mu_{-i}, a_\ell)] = d(\mu_i)F_{\mu_{-i}}(\mu_i),$$

where $d(\mu_i) \equiv \left[p_2 + \frac{\partial}{\partial \mu_i} c\left(\frac{c^{-1}(p_2(1-\mu_i))}{a_h}\right)\right]$. The distribution function of beliefs in the second contest depends on the distribution function of output in the first contest. The convexity of the cost of effort implies that $d(\mu_i) \in \left[\frac{p_2(a_h-1)}{a_h}, p_2\right)$ for all $\mu_i \in (0, 1)$, guaranteeing that these incentives are strict.

4 First contest

We now turn to the analysis of the first contest. Prior to this contest, the contestants are ex-ante symmetric with probability of being high ability, $\hat{\mu}$. They become privately informed of their ability prior to choosing output in the first contest. Additionally, the contestants are forward looking, understanding that their output choice in the first contest will be revealed to the other contestant and subsequently will affect that contestant's beliefs and second contest strategies as analyzed in the previous section.

For each player $i = 1, 2$, $L_1^i(x)$ and $H_1^i(x)$ denote the first period output distributions of a contestant with low ability and high ability respectively. Then the ex-ante expected output distribution is $F_1^i(x_1) = (1-\hat{\mu})L_1^i(x_1) + \hat{\mu}H_1^i(x_1)$, for $i = 1, 2$. Additionally, f_1^i, ℓ_1^i and h_1^i denote the densities that are induced from the distribution functions F_1^i, L_1^i and H_1^i.[10] Lastly, define $X_1^{h,i} = \{x|h_1^i(x) > 0\}$ and $X_1^{\ell,i} = \{x|\ell_1^i(x) > 0\}$.

We restrict attention to equilibria that are symmetric. A set of output distributions $\{H_1^i(x_1), L_1^i(x_1), H_2^i(x_2|\mu_i, \mu_{-i}), L_2^i(x_2|\mu_i, \mu_{-i})$ for $i = 1, 2\}$ form a symmetric perfect Bayesian equilibrium (SPBE) for two successive contests if

1. output distributions in the first contest are symmetric: $H_1^1(x) = H_1^2(x)$, and $L_1^1(x) = L_1^2(x)$,

2. contestants play the unique Bayes Nash equilibrium in the second contest: for $i = 1, 2$ and for every (μ_i, μ_{-i})

$$(H_2^i(x|\mu_i, \mu_{-i}), L_2^i(x|\mu_i, \mu_{-i})) = \begin{cases} (H_w^*(x|\mu_i, \mu_{-i}), L_w^*(x|\mu_i, \mu_{-i})), & \text{if } \mu_i \leq \mu_{-i} \\ (H_s^*(x|\mu_i, \mu_{-i}), L_s^*(x|\mu_i, \mu_{-i})), & \text{if } \mu_i > \mu_{-i} \end{cases},$$

3. players update beliefs according to Bayes rule when feasible:[11]

$$\mu_i = \mu(x_1^i) = \frac{h_1(x_1^i)}{h_1(x_1^i) + \ell_1(x_1^i)}, \text{ for } i = 1, 2, \text{ and}$$

4. given (2) and (3), contestants always choose an optimal output in the first contest:

[10] The extended definition of density using Dirac-delta functions is invoked where necessary.

[11] Using the extended definition of density allows agents to update their beliefs even when they see their opponent produce an output where the distribution has a mass point. For example, if the H_1 has a mass point at x, while L_1 does not, this definition implies $\mu(x_1) = 1$.

for $i = 1, 2$,

$$X_1^{\ell,i} \subseteq \arg\max_{x_1^i} \mathbb{E}[\pi(x_1^i, x_2^i(\mu(x_1^i), \mu(x_1^{-i}), a_\ell), a_\ell)] \equiv BR_1(a_\ell)$$
$$\text{and } X_1^{h,i} \subseteq \arg\max_{x_1^i} \mathbb{E}[\pi(x_1^i, x_2^i(\mu(x_1^i), \mu(x_1^{-i}), a_h), a_h)] \equiv BR_1(a_h).$$

Given conditions (1)-(4), the payoffs in the second contest for a given $\mu_i(x_1^i)$ and $\mu_{-i}(x_1^{-i})$ are characterized by $v_i(\mu(x_1^i), \mu(x_1^{-i}), a^i)$ as in Corollary 1, where $i = s$ if $\mu(x_1^i) \geq \mu(x_1^{-i})$ and $i = w$ otherwise. Therefore, the expected payoffs to player i for the two contests can be written in terms of output in the first contest.

$$\mathbb{E}[\pi^i(x_1^i, \hat{x}_2^i(\mu(x_1^i), \mu(x_1^{-i}), a^i), a^i)] = \mathbb{E}[\pi_1^i(x_1^i, a^i)] + \mathbb{E}[v_i(\mu(x_1^i), \mu(x_1^{-i}), a^i)]$$

For a given expected output distribution, $F_1(x)$, the indifference conditions for all outputs within the best response sets of the high and low ability contestants are respectively

$$BR(a_h) : p_1 F_1^*(x) + \mathbb{E}[v_i(\mu(x), \mu(x^{-i}), a_h)] - c(x/a_h) = K_h(p_1, p_2) \text{ and}$$
$$BR(a_\ell) : p_1 F_1^*(x) + \mathbb{E}[v_i(\mu(x), \mu(x^{-i}), a_\ell)] - c(x) = K_\ell(p_1, p_2),$$

where $K_h(p_1, p_2)$ and $K_\ell(p_1, p_2)$ are the expected payoffs of high and low ability contestants in the repeated contest game. Ignoring the continuation value of the second contest, these indifference conditions would lead to a monotonic output distribution as in the second contest. However, the presence of the second contest continuation value creates a countervailing incentive. The contestant with high ability has an incentive to reduce output in the first contest, i.e. sandbag, in order to appear weaker in the second contest while the low ability contestant that would prefer to look strong entering the second contest by choosing high output, i.e. bluff. Despite these confounding effects, a variation of the monotonically condition persists. In equilibrium, higher output in the first contest cannot lead to a lower belief about the contestant's ability.

Lemma 2. *In every SPBE, $\mu(x)$ is weakly increasing in x for all $x \in X_1 = X_1^h \cup X_1^\ell$.*

In addition to restricting the belief function on the equilibrium path, the countervailing incentives also determine the beliefs that can be assumed for outputs that are not chosen in equilibrium. In games where players can signal private information, there are often equilibria where actions off the equilibrium path are not taken because players who do so are assumed to have negative characteristics.[12] However, any off-path

[12] See for example Spence (1973).

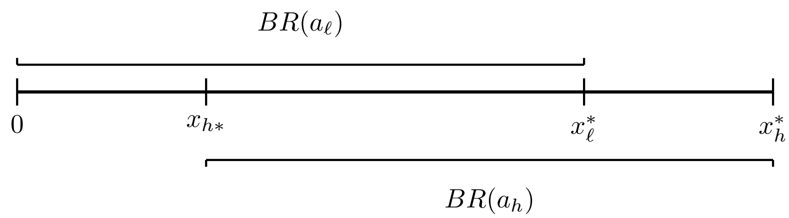


Figure 2: Representation of best response sets of the high ability and low ability contestants in the first contest.

belief will benefit at least one type of contestant due to their opposing incentives. Because the belief function must be well behaved for all potential outputs, the first period equilibrium strategies conform to nice properties without additional refinements. In particular, equilibrium strategies have no atoms, there are no gaps in the corresponding best response sets and payoffs in the first contest are continuous in output. The countervailing incentives also guarantee that the intersection of the best response sets for the low ability and high ability contestants is non-trivial. In any equilibrium, the best response sets of the high and low ability contestants partition outputs of the first contest into at most three distinct intervals. The endpoints of the intervals are determined by the sets where the density of low and high ability distributions are positive. Specifically, define $x_{\ell *} = \inf X_1^\ell$, $x_\ell^* = \sup X_1^\ell$, $x_{h*} = \inf X_1^h$, and $x_h^* = \sup X_1^h$. These intervals and best response sets are shown in Figure 2.

Lemma 3. *There is no output that is played with positive probability and* $\Pr(win|x) = F_1(x)$ *is continuous.*

Lemma 4. *The best response sets of low and high ability contestants are intervals with* $BR(a_\ell) = [0, x_\ell^*]$, $BR(a_h) = [x_{h*}, x_h^*]$ *and* $0 = x_{\ell *} \leq x_{h*} < x_\ell^* \leq x_h^*$.

From the above properties, it follows that the belief function $\mu(x)$ must be continuous and weakly increasing on the union of the best response sets, $[0, x_h^*]$. This argument and its implications on the belief function are formalized in Lemma 5.

Lemma 5. *The belief function and the distribution functions of output,* $L_1(x)$ *and* $H_1(x)$*, are continuous in output on* $[0, x_h^*]$*. Additionally, the belief function is weakly increasing on* (x_{h*}, x_ℓ^*)*, takes a value of zero for all* $x \in [0, x_{h*}]$ *when* $x_{h*} > 0$*, and takes a value of one for all* $x \in [x_\ell^*, x_h^*]$ *when* $x_h^* > x_\ell^*$.

The above lemmas allow complete characterization of the output distribution, $F_1(x)$ on $[0, x_h^*]$. Our main result shows that this characterization is unique and implies the uniqueness of the equilibrium belief function and the output distributions of both high and low ability contestants.

Theorem 1. *There is a unique symmetric perfect Bayes Nash equilibrium that satisfy conditions (1)-(4),* $\{(L_1^*(x_1), L_2^*(x_2|\mu_i, \mu_{-i})), (H_1^*(x_1), H_2^*(x_2|\mu_i, \mu_{-i}))\}$.

The ex-ante output distribution for each contestant in the first contest can be characterized in each of the three intervals.

All outputs in the range $x_{h*} \leq x \leq x_\ell^*$ must be in the best response set of both low and high ability contestants. Because the marginal cost of increasing output for the low ability contestant is always more than for the high ability contestant, the marginal benefit must also be higher for the low ability contestant. Since the benefit of increasing output is the same for each type in the first contest, this difference must come from the second contest continuation values. The difference in marginal benefit of appearing stronger in the second contest must equal the difference in the marginal cost of output in the first contest. This leads to the condition in Equation 1.

$$\mu'(x)d(\mu(x)) = c'(x) - \frac{1}{a_h}c'\left(\frac{x}{a_h}\right) \tag{1}$$

Combining this with the best response conditions of both the low and high ability contestants results in the differential equation that the output density function must satisfy over this interval (Equation 2). For a given value of x_{h*}, this differential equation has a unique solution.

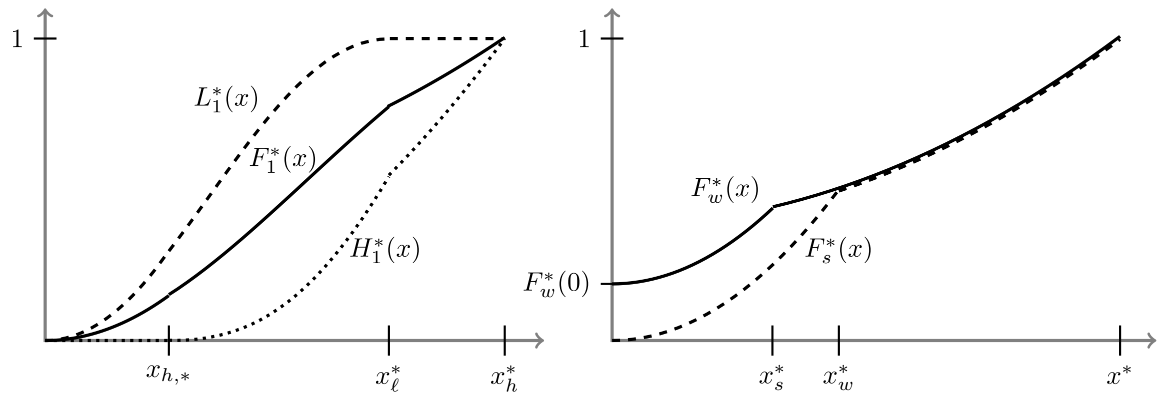
$$p_1 f_1^*(x) = c'(x)(1 - F_1^*(x)) + \frac{1}{a_h}c'\left(\frac{x}{a_h}\right)F_1^*(x) \tag{2}$$

For outputs in the range of $0 \leq x < x_{h*}$, Lemma 5 states $\mu(x) = 0$ and therefore $\mathbb{E}[v_i(\mu(x), \mu(x^{-i}), a_\ell)] = 0$. Since $x = 0$ is in $BR(a_\ell)$, $K_\ell(p_1, p_2) = 0$ in any equilibrium. From the condition for $BR(a_\ell)$, the distribution is $F_1^*(x) = \frac{1}{p_1}c(x)$.

Similarly, for outputs in the range $x_\ell^* < x \leq x_h^*$, $\mu(x) = 1$ and the distribution function is $F_1^*(x) = \frac{1}{p_1}(c(x/a_h) + K_h(p_1, p_2) - v_h)$, where $v_h = \mathbb{E}[v_i(1, \mu(x^{-i}), a_h)]$, the expected payoff in the second contest for a contestant who is known to have high ability.

5 Welfare

The unique equilibrium can be constructed when the cost function is homogeneous. Specifically, for this section we let $c(x) = kx^\alpha$, with $\alpha \geq 1$ and $k > 0$. Note that this includes the case that is isomorphic to the all-pay auction. Additionally, we take $\hat{\mu} = 1/2$, so that contestants are equally likely to be high or low ability. Construction of the equilibrium is given in appendix B. Figure 3 gives an example of the equilibrium



(a) First Contest

(b) Second Contest

Figure 3: (a) High ability, low ability and expected output distributions in the first contest with $c(e) = e^2$ and $\hat{\mu} = 1/2$; (b) Expected output distributions of the strong and weak contestants in the second contest with $c(e) = e^2$ and $\hat{\mu} = 1/2$.

output distributions with quadratic cost of effort. This construction allows for the characterization of expected payoffs and output.

Proposition 3. *The expected payoff for a low ability contestant is zero for any prize allocation over two contests. The expected payoff of a high ability contestant is*

$$K_h(p_1, p_2) = \frac{(a_h^\alpha - 1)\,(c(x_{h*}) + p_2\,(1 - \mu(x_{h*})))}{a_h^\alpha}.$$

While the expected output in the first contest is determined directly by $F_1^*(x)$, the expected output in the second contest depends on the beliefs about each contestant. Therefore the ex-ante expected output depends on the frequency that each set of beliefs is realized from output choices in the first contest. The equilibrium distribution of beliefs, $F_\mu(M) = \Pr(\mu \leq M)$, follows the expected output distribution for $x \in (x_{h*}, x_\ell^*)$ as the belief function is strictly increasing over this interval. Specifically, $F_\mu(M) = F_1^*(\mu^{-1}(M))$ for this interval of outputs. Moreover, $F_\mu(0) = F_1^*(x_{h*})$, $F_\mu(1) = 1$, and $\lim_{M \to 1} F_\mu(M) = 1 - F_1^*(x_\ell^*)$.

Ex-ante expected total output over the two contests is

$$2\int_0^{x_h^*} x dF_1^*(x) + \int_0^1 \int_0^1 \mathbb{E}[x_s(\mu_s, \mu_w, a^s) + x_w(\mu_w, \mu_s, a^w)] dF_\mu(\mu_i) dF_\mu(\mu_j),$$

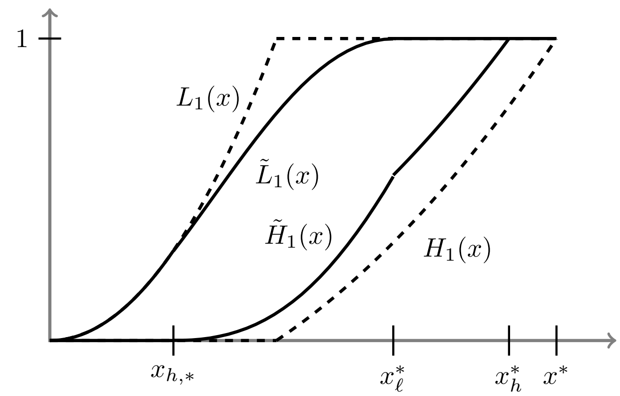


Figure 4: Distribution of strategies in the first of two contests with no second contest prize (dashed) and with even prizes in each contest.

where $\mu_s = \max\{\mu_i, \mu_j\}$ and $\mu_w = \min\{\mu_i, \mu_j\}$ and $\mathbb{E}[x_s(\mu_s, \mu_w, a^s) + x_w(\mu_w, \mu_s, a^w)]$ is

$$\left(\frac{\alpha}{\alpha+1}\right)\left(\frac{p_2}{k}\right)^{1/\alpha}\left[\left(\frac{a_h^\alpha - 1}{a_h^\alpha}\right)\left((1-\mu_w)^{\frac{\alpha+1}{\alpha}} + (1-\mu_s)^{\frac{\alpha+1}{\alpha}}\right) + 2a_h\left(\mu_w + \frac{1-\mu_w}{a_h^\alpha}\right)^{\frac{\alpha+1}{\alpha}}\right].$$

When one contestant enters the second contest in a significantly stronger position, the overall output in this contest is reduced. The weaker contestant faces a negative motivation effect while the stronger contestant will not compete as aggressively against a weak opponent. Specifically, for a fixed value of μ_w, an increase in μ_s leads to a reduction in expected aggregate output in the second contest.

$$\frac{\partial}{\partial \mu_s}\mathbb{E}[x_s(\mu_s, \mu_w, a^s) + x_w(\mu_w, \mu_s, a^w)] = -\left(\frac{p_2}{k}\right)^{1/\alpha}\left(\frac{a_h^\alpha - 1}{a_h^\alpha}\right)(1-\mu_s)^{\frac{1}{\alpha}} < 0.$$

5.1 Prize Allocation

The prizes in each contest impact the strategies of the first contest and the information available in the second contest. A change in p_1 impacts the marginal benefit of output in the first contest while a change in p_2 impacts the continuation values for a given belief entering the second contest. Increasing the prize in the first contest increases the incentive for high ability contestants to separate from low ability contestants. As a result, public beliefs about contestants' abilities are on average more separated entering the second contest. A larger prize in the second contest increases the incentive of high ability contestants to sandbag and low ability contestants to bluff. This is depicted in Figure 4. Additional pooling in the first contest on average reveals less information about opponents' abilities prior to the second contest.

Proposition 4. *Let $F_\mu(M)$ be the belief distribution associated with prize ratio p_1/p_2 and $\tilde{F}_\mu(M)$ be associated with $\tilde{p}_1/\tilde{p}_2$. Then $p_1/p_2 > \tilde{p}_1/\tilde{p}_2$ implies $F_\mu(M) <_{SOSD} \tilde{F}_\mu$.*

The ratio of prizes in the two contests impact both the efficiency and output of the two contests. When the first contest has large stakes relative to the second contest, the winner of the first contest is more likely to be high ability and the loser of the second contest is more likely to be low ability. As a result, the second contest will be, on average, more asymmetric in terms of beliefs about contestants' expected abilities.

Corollary 2. *A higher prize ratio, p_1/p_2, decreases the probability of that a low ability player wins the first contest against a high ability player, increases the expected belief about the stronger contestant and decreases the expected belief about weaker contestant in the second contest.*

Fixing the total prize, which we normalize to one, we numerically investigate the impact of how prizes are allocated between the first and second contests. Allocating a relatively larger prize to the second contest increases the expected payoffs of the contestants, see Figure 5. Following from Proposition 4, we know that there is stronger incentives to pool when the second contest has relatively high stakes. The pooling strategies, bluffing and sandbagging, are complementary: an increased incentive for the low ability to appear strong makes it easier to the high ability contestant to appear weak. Therefore, the high ability contestants are able to attain a more favorable position in the second contest at a lower cost, i.e. a smaller deviation from choosing one period optimal effort in the first contest.

Total expected output over the two contests is impacted both by the partial pooling strategies in the first contest and the information available in the second contest. While sandbagging and bluffing have opposite impacts in the first contest, less pooling in the first contest increases the likelihood of a less competitive second contest with lower output. Expected output for linear cost of effort and convex cost of effort is given in Figure 6.

When cost of effort is linear, allocating the entire prize purse to a single contest will maximize the expected output. When effort costs in each contest are convex, output is increased when the prize purse is spread over two contests. An asymmetric allocation of prizes avoids a second contest with both a large prize and low level of competition. Allocating a majority of the prize to the first contest minimizes the impact of lower competition in the second contest, while allocating a majority of the prize to the second contest reduces the likelihood of lessened competition in the second contest by increasing pooling in the first contest. Numerical simulation suggests two local maxima for these

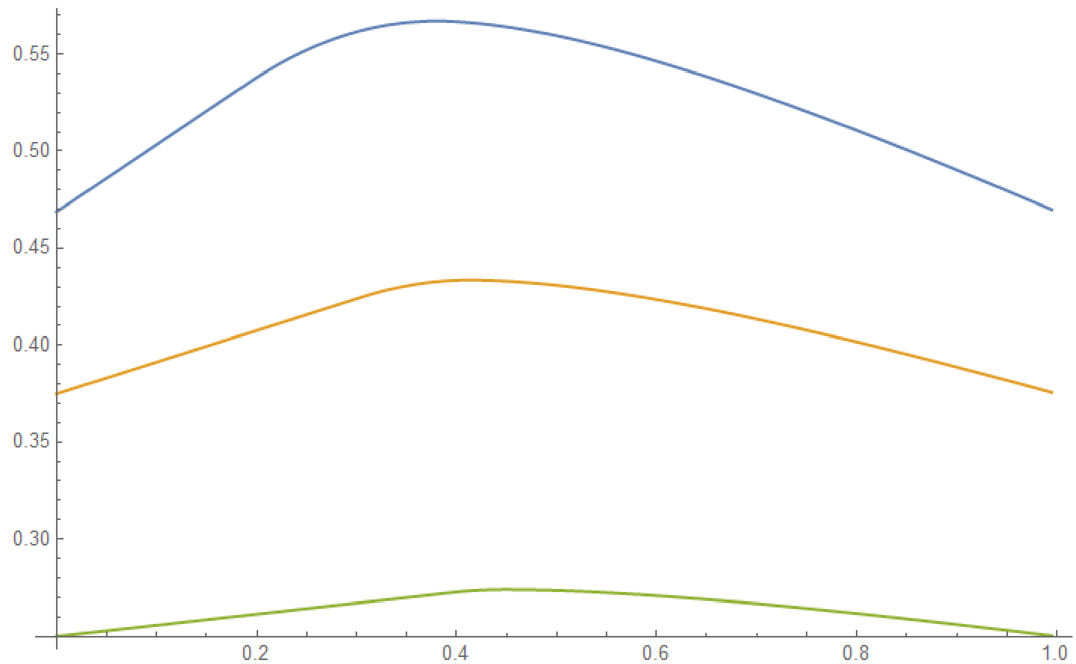


Figure 5: Expected payoffs with $a_h = 2$ and cost of effort: $c(e) = ke$ (green), $c(e) = ke^2$ (yellow) and $c(e) = ke^4$ (blue).

two asymmetric allocations of prizes with convex costs. To maximize "excitement" contest designers can have a low stakes initial round followed by high stakes final round. If identification of a high ability contestant is more important, then a high stakes initial competition (choosing a leader) can be followed by lower stakes to continue encouraging effort (sticking by the leader almost certainly).

Expected payoffs are equal to the total prize purse less the expected cost of effort. All else equal, as expected payoffs increase the total expected output would decrease. However, if more effort is allocated to higher ability contestants, then the same cost of effort leads to higher output. Therefore strategies with less pooling will be more efficient. Moreover, when facing convex costs, uneven prizes will tend to increase total effort costs for a given output level.

6 Conclusion

When private information about contestants' abilities can be revealed, contestants have an incentive to signal-jam. Low ability contestants prefer that their opponents believe they are high ability while high ability contestants prefer the opposite. In the course of a dynamic competition these incentives lead low ability contestants to bluff by ex-

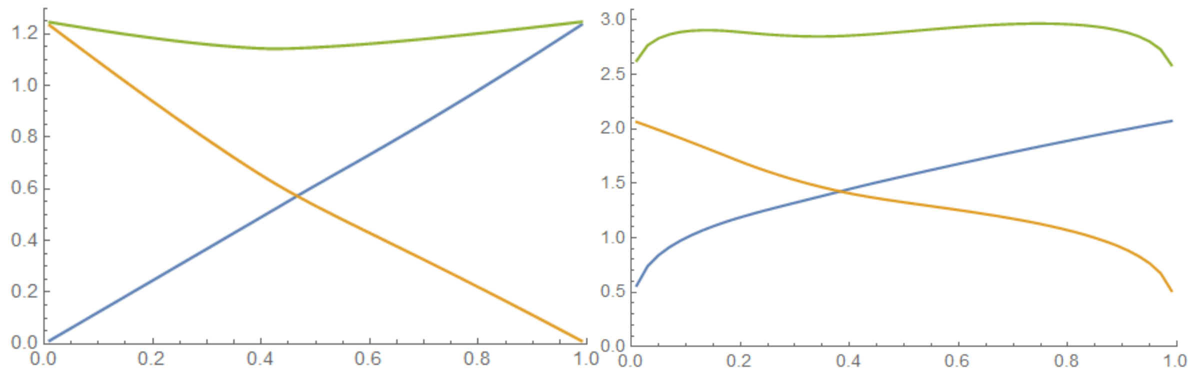


Figure 6: Expected total output (green), second contest output (yellow), and first contest output (blue) against percentage of total prize allocated to first contest with $a_h = 2$ and $c(e) = e$ (left) and $c(e) = e^4$ (right).

erting more effort and high ability contestants to sandbag by exerting less effort in the initial stages of the competition. Initial strategies impact the information available to contestants in later stages of the contest. Bluffing and sandbagging reduce the informativeness of contestants' outputs, lowering the chance of creating asymmetries in the later stages of the contest that can reduce competition.

Preventing the release of information in a dynamic contest can alleviate inefficiencies caused by signal-jamming and information induced asymmetries. When information cannot be withheld, the relative stakes before and after information is released can be adjusted to control the impact of the information on the contest. Larger stakes prior to information release will increase the chance that the best contestant wins the initial contest. Larger stakes after will increase the chance that the second contest is competitive.

Appendix

A Proofs

Proof of Lemma 1

The proof follows in three steps:

(1) There is no output at which both contestants have an atom. If a contestant has an atom, it is at zero.

Assume both contestants produce x with positive probability. Because the cost of effort is continuous, either contestant can improve payoffs by producing output slightly above this atom. Then x is not a best response of that contestant, a contradiction.

Assume that contestant i produces $x > 0$ with positive probability. Then by the continuity of the cost function in output, there is a $\delta > 0$ such that $\hat{x} \in (x - \delta, x)$, $\hat{x} \notin BR_{-i}(a^{-i})$. This implies, that contestant i would do better by playing $x - \delta/2$, and therefore $x \notin BR_i(a^i)$, a contradiction. Therefore the ex-interim output distribution functions of each contestant are continuous on $(0, \infty)$.

(2) If $\hat{x} > 0$ is not a best response for any ability of one of the contestants, then for all $x > \hat{x}$, x is not a best response for either type of either contestant.

Step (1) implies that payoffs are continuous. Given $x \notin \{BR_i(a_\ell) \cup BR_i(a_h)\}$, for some $i = s, w$, $\exists \tilde{x}(a_h), \tilde{x}(a_\ell)$ for which $\pi^i(\tilde{x}(a_h), a_h) > \pi^i(x, a_h) + \varepsilon$ and $\pi^i(\tilde{x}(a_\ell), a_\ell) > \pi^i(x, a_\ell) + \varepsilon$. Then, every x in this neighborhood cannot be a best response of either type of contestant i, and therefore cannot be a best response for any type of contestant $-i$, who could improve expected payoffs by lowering output. Now let X^* be the set of all outputs that are greater than $\hat{x}$ and are a best response for some contestant of either type. Let $x_* = \inf\{X^*\}$. Then, there is a neighborhood below x_* for which there are no best responses. This would imply however, that there is an $x \in X^*$ that gives lower expected payoffs than $\hat{x}$ for both types of each contestant, a contradiction. Therefore, x_* does not exist and X^* is empty. This implies that $\sup\{BR_s(a_\ell) \cup BR_s(a_h)\} = \sup\{BR_w(a_\ell) \cup BR_w(a_h)\} \equiv x^*$ and the combined best response sets of each contestant is $[0, x^*]$.

(3) Any equilibrium must be monotonic i.e., for $x \in BR_i(a_\ell)$ and $x' \in BR_i(a_h)$, then $x \leq x'$.

Assume otherwise, $\exists x \in BR_i(a_\ell)$ and $x' \in BR_i(a_h)$, with $x > x'$. Then

$$\mathbb{E}[\pi^i(x, 1) - \pi^i(x', 1) = p_2(F_{-i}(x) - F_{-i}(x')) - (c(x) - c(x')) \geq 0.$$

Because the cost function is increasing and weakly convex, and $x > x'$, then

$$\mathbb{E}[\pi^i(x, a_h) - \pi^i(x', a_h) = p_2(F_{-i}(x) - F_{-i}(x')) - (c(x/a_h) - c(x'/a_h)) > 0.$$

This contradicts $x' \in BR_i(a_h)$.

Proof of Proposition 1

From Lemma 1, there are x_s^* and x_w^* such that $x_i^* = \sup\{BR_i(a_\ell)\} = \inf\{BR_i(a_h)\}$, for $i = s, w$. We also define $x^* \equiv \sup\{BR_i(a_h)\}$ which is the same for $i = s, w$.

Each contestant must be indifferent between all $x \in (x_i^*, x^*)$ when they have high ability. Each high ability contestant has the same marginal cost of output so indifference implies that the expected output density must also be the same: $f_s^*(x) = f_w^*(x)$ for $x \in (\max\{x_s^*, x_w^*\}, x^*)$. Since $f_i^*(x) = \mu_i h_i^*(x)$ for all $x \in (x_i^*, x^*)$, then $h_s^*(x) \leq h_w^*(x)$ for all $x \in (\max\{x_s^*, x_w^*\}, x^*)$. Then $H_i(x_i^*) = 0$, requires that $x_s^* \leq x_w^*$.

Therefore, for the remainder of the construction, there are three intervals to consider: the best response set of the low types of both the stronger and the weaker contestants, $0 \leq x \leq x_s^*$, the best response set of the low type of the weaker contestant and the high type of the strong contestant, $x_s^* \leq x \leq x_w^*$, and best response set of the high types of each contestant, $x_w^* \leq x \leq x^*$.

Within their best response sets, contestants must be indifferent between all output levels. For example, the strong contestant with high ability must be indifferent to picking all outputs between x_s^* and x^*. This puts a condition on $F_w(x)$, the output distribution of the weak contestant, on the interval $[x_s^*, x^*]$:

$$p_2 F_w^*(x') - c\left(\frac{x}{a_h}\right) = p_2 F_w^*(x') - c\left(\frac{x'}{a_h}\right).$$

Rearranging and taking the limit as $x \to x'$, $\lim_{x \to x'} \frac{F_w^*(x) - F_w^*(x')}{c(\frac{x}{a_h}) - c(\frac{x'}{a_h})} = \frac{1}{p_2}$. Then the output density of the strong contestant is

$$f_w^*(x') = \lim_{x \to x'} \frac{F_w^*(x) - F_w^*(x')}{c(\frac{x}{a_h}) - c(\frac{x'}{a_h})} \frac{c(\frac{x}{a_h}) - c(\frac{x'}{a_h})}{a_h(\frac{1}{a_h}(x - x'))} = \frac{c'(x'/a_h)}{p_2 a_h}.$$

A similar calculation on each interval for each contestant allows us to characterize the densities of the output on each of the intervals below.

- $x_w^* \leq x \leq x^*$: $h_s^*(x) = \frac{c'(x/a_h)}{p_2 a_h \mu_s}$, $h_w^*(x) = \frac{c'(x/a_h)}{p_2 a_h \mu_w}$, $f_s^*(x) = f_w^*(x) = \frac{c'(x/a_h)}{p_2 a_h}$.
- $x_s^* \leq x \leq x_w^*$: $h_s^*(x) = \frac{c'(x)}{p_2 \mu_s}$, $\ell_w^*(x) = \frac{c'(x/a_h)}{p_2 a_h (1-\mu_w)}$, $f_s^*(x) = \frac{c'(x)}{p_2}$, $f_w^*(x) = \frac{c'(x/a_h)}{p_2 a_h}$.
- $0 \leq x \leq x_s^*$: $\ell_s^*(x) = \frac{c'(x)}{p_2(1-\mu_s)}$, $\ell_w^*(x) = \frac{c'(x)}{p_2(1-\mu_w)}$, $f_s^*(x) = f_w^*(x) = \frac{c'(x)}{p_2}$.

It remains to characterize the cutoff points, x_w^*, x_s^* and x^*, and $L_w(0)$. In equilibrium, the distribution of output for each contestant must satisfy

$$L_i^*(x_i^*) = 1, \quad H_i^*(x_i^*) = 0, \quad F_i^*(x_i^*) = 1 - \mu_i, \text{ and } \quad F_i^*(x^*) = 1.$$

Additionally, the strong contestant chooses no effort with zero probability, so $L_s^*(0) = 0$. Using $L_s^*(x_s^*) = 1$ and the definition of $\ell_s^*(x)$ on $[0, x_s^*]$, we calculate x_s^*.

$$\int_0^{x_s^*} \ell_s^*(x) dx = L_s^*(x_s^*) - L_s(0) = \frac{c(x_s^*)}{p_2(1 - \mu_s)} = 1$$

Then $c(x_s^*) = p_2(1-\mu_s)$, so that $x_s^* = c^{-1}(p_2(1-\mu_s))$. Similarly, $x_w^* = c^{-1}(p_2(1-\mu_w))$. From these endpoints we can calculate x^*.

$$\int_{x_s^*}^{x_w^*} h_s^*(x)dx = \frac{c(x_w^*) - c(x_s^*)}{\mu_s} = \frac{(1-\mu_w) - (1-\mu_s)}{\mu_s} = \frac{\mu_s - \mu_w}{\mu_s}$$

$$\int_{x_w^*}^{x^*} h_s^*(x_s)dx = 1 - \frac{\mu_s - \mu_w}{\mu_s} = \frac{\mu_w}{\mu_s}$$

$$\int_{x_w^*}^{x^*} f_s^*(x_s)dx = \frac{1}{p_2}\left(c\left(\frac{x^*}{a_h}\right) - c\left(\frac{c^{-1}(p_2(1-\mu_w))}{a_h}\right)\right) = \mu_w$$

$$x^* = a_h c^{-1}\left(p_2\mu_w + c\left(\frac{c^{-1}(p_2(1-\mu_w))}{a_h}\right)\right)$$

Lastly, we pin down the probability that the weaker contestant exerts no effort.

$$\int_{x_s^*}^{x_w^*} \ell_w^*(x)dx = \frac{1}{p_2(1-\mu_w)}\left[c\left(\frac{c^{-1}(p_2(1-\mu_w))}{a_h}\right) - c\left(\frac{c^{-1}(p_2(1-\mu_s))}{a_h}\right)\right]$$

$$\int_{0}^{x_s^*} \ell_w^*(x)dx = \frac{c(c^{-1}(p_2(1-\mu_s)))}{p_2(1-\mu_w)} - 0 = \frac{1-\mu_s}{1-\mu_w}$$

$$L_w^*(0) = \mu_s - \mu_w - \frac{1}{p_2(1-\mu_w)}\left[c\left(\frac{c^{-1}(p_2(1-\mu_w))}{a_h}\right) - c\left(\frac{c^{-1}(p_2(1-\mu_s))}{a_h}\right)\right]$$

Proof of Corollary 1

The expected payoffs of a high ability contestant are equal to the value of winning less the cost of producing output x^*, as producing x^* guarantees a win.

$$v_s(\mu_s, \mu_w, a_h) = v_w(\mu_w, \mu_s, a_h) = p_2 - c(x^*/a_h) = p_2(1-\mu_w) - c\left(\frac{c^{-1}(p_2(1-\mu_w))}{a_h}\right)$$

The expected payoffs of low ability contestants is equal to the probability they win given they exert no effort. This is the probability the other contestant puts in no effort.[13]

$$\begin{aligned} v_s(\mu_s, \mu_w, a_\ell) &= p_2(1-\mu_w)L_w(0) \\ &= p_2(\mu_s - \mu_w) - \left[c\left(\frac{c^{-1}(p_2(1-\mu_w))}{a_h}\right) - c\left(\frac{c^{-1}(p_2(1-\mu_s))}{a_h}\right)\right] \\ v_w(\mu_w, \mu_s, a_\ell) &= p_2(1-\mu_s)L_s(0) = 0 \end{aligned}$$

[13] Here we technically are assuming the contestant wins all ties at zero, but if the agent exerts a tiny amount of effort and we let that effort shrink to zero, then this is the payoff of the agent in the limit. Since the payoffs are continuous, these limits must be the payoffs of the low ability contestants.

Proof of Proposition 2

In the second contest, for a given pair of beliefs, contestants will expect the following payoffs:

$$v_i(\mu_i, \mu_{-i}, a_h) = p_2(1 - \min\{\mu_i, \mu_{-i}\}) - c\left(\frac{c^{-1}(p_2(1 - \min\{\mu_i, \mu_{-i}\}))}{a_h}\right)$$

$$v_i(\mu_i, \mu_{-i}, a_\ell) = \begin{cases} p_2(\mu_i - \mu_{-i}) - \left[c\left(\frac{c^{-1}(p_2(1-\mu_{-i}))}{a_h}\right) - c\left(\frac{c^{-1}(p_2(1-\mu_i))}{a_h}\right)\right], & \text{if } \mu_i \geq \mu_{-1} \\ 0, & \text{otherwise} \end{cases}$$

For a high ability contestant believed to be high ability with probability μ_i and with opponent's belief distribution, $F_{\mu_{-i}}$, the expected payoff in the second contest is

$$\begin{aligned} &\mathbb{E}[v_i(\mu_i, \mu_{-i}, a_h)] \\ &= \int_0^1 \left(p_2(1 - \min\{\mu_i, \mu_{-i}\}) - c\left(\frac{c^{-1}(p_2(1 - \min\{\mu_i, \mu_{-i}\}))}{a_h}\right)\right) dF_{\mu_{-i}}(\mu_{-i}) \end{aligned}$$

As the opponent believes the contestant is stronger, the change in expected payoff is

$$\frac{\partial}{\partial \mu_i} E_{\mu_{-i}}[v_i(\mu_i, \mu_{-i}, a_h)] = \left(p_2 + \frac{\partial}{\partial \mu_i} c\left(\frac{c^{-1}(p_2(1 - \mu_i))}{a_h}\right)\right)(F_{\mu_{-i}}(\mu_i) - 1).$$

For a low ability contestant, the expected payoff is

$$\begin{aligned} &\mathbb{E}[v_i(\mu_i, \mu_{-i}, a_\ell)] \\ &= \int_0^{\mu_i} \left(p_2(\mu_i - \mu_{-i}) + c\left(\frac{c^{-1}(p_2(1 - \mu_i))}{a_h}\right) - c\left(\frac{c^{-1}(p_2(1 - \mu_{-i}))}{a_h}\right)\right) dF_{\mu_{-i}}(\mu_{-i}), \end{aligned}$$

with change in expected payoff

$$\frac{\partial}{\partial \mu_i} \mathbb{E}[v_i(\mu_i, \mu_{-i}, a_\ell)] = \left(p_2 + \frac{\partial}{\partial \mu_i} c\left(\frac{c^{-1}(p_2(1 - \mu_i))}{a_h}\right)\right) F_{\mu_{-i}}(\mu_i).$$

Given the assumptions on the cost of effort, $c'(e) > 0$ and $c''(e) \geq 0$,

$$\frac{\partial}{\partial \mu_i} c\left(\frac{c^{-1}(p_2(1 - \mu_i))}{a_h}\right) = -\frac{1}{a_h} c'\left(\frac{c^{-1}(p_2(1 - \mu_i))}{a_h}\right) \frac{1}{c'(c^{-1}(p_2(1 - \mu_i)))} \in \left[-\frac{p_2}{a_h}, 0\right).$$

Define $d(\mu_i) \equiv \left[p_2 + \frac{\partial}{\partial \mu_i} c\left(\frac{c^{-1}(p_2(1-\mu_i))}{a_h}\right)\right]$. For all μ_i, $d(\mu_i) \in \left[\frac{p_2(a_h - 1)}{a_h}, p_2\right)$. It follows that

$$\frac{\partial}{\partial \mu_i} E[v_i(\mu_i, \mu_{-i}, a_h)] = d(\mu_i)(F_{\mu_{-i}}(\mu_i) - 1) \text{ and } \frac{\partial}{\partial \mu_i} E[v_i(\mu_i, \mu_{-i}, a_\ell)] = d(\mu_i) F_{\mu_{-i}}(\mu_i),$$

where each derivative is strictly positive when $\mu_i \in (\underline{M}_{-i}, \overline{M}_{-i})$.

Proof of Lemma 2

Assume that for a given x and y which are best responses for some ability level we have that $x < x'$ and $\mu(x) > \mu(x')$. Then $0 \leq \mu(x') < \mu(x) \leq 1$, $h_1(x) > 0$ and $\ell_1(x') > 0$. Then equilibrium requires $x \in BR(a_h)$ and $x' \in BR(a_\ell)$ and therefore

$$\begin{aligned} &p_1 \Pr(\text{win}|x') - c(x') + \mathbb{E}[v_i(\mu(x'), \mu(x^{-i}), a_\ell)] \\ &\qquad \geq p_1 \Pr(\text{win}|x) - c(x) + \mathbb{E}[v_i(\mu(x), \mu(x^{-i}), a_\ell)], \text{ and} \\ &p_1 \Pr(\text{win}|x') - c(x'/a_h) + \mathbb{E}[v_i(\mu(x'), \mu(x^{-i}), a_h)] \\ &\qquad \leq p_1 \Pr(\text{win}|x) - c(x/a_h) + \mathbb{E}[v_i(\mu_i(x), \mu(x^{-i}), a_h)]. \end{aligned}$$

This implies that

$$\begin{aligned} &p_1(\Pr(\text{win}|x') - \Pr(\text{win}|x)) + \mathbb{E}[v_i(\mu(x'), \mu(x^{-i}), a_\ell)] - \mathbb{E}[v_i(\mu(x), \mu(x^{-i}), a_\ell)] \\ &\qquad \geq c(x') - c(x), \text{ and} \\ &p_1(\Pr(\text{win}|x') - \Pr(\text{win}|x)) + \mathbb{E}[v_i(\mu(x'), \mu(x^{-i}), a_h)] - \mathbb{E}[v(\mu(x), \mu(x^{-i}), a_h)] \\ &\qquad \leq c(x'/a_h) - c(x/a_h). \end{aligned}$$

From Proposition 2, $\mu(x) > \mu(x')$ implies

$$\begin{aligned} \mathbb{E}[v_i(\mu(x'), \mu(x^{-i}), a_\ell)] - \mathbb{E}[v_i(\mu(x), \mu(x^{-i}), a_\ell)] \geq 0, \\ \text{and } \mathbb{E}[v_i(\mu(x'), \mu(x^{-i}), a_h)] - \mathbb{E}[v(\mu(x), \mu(x^{-i}), a_h)] \geq 0. \end{aligned}$$

Combining the previous inequalities,

$$c(x') - c(x) \leq p_1(\Pr(\text{win}|x') - \Pr(\text{win}|x)) \leq c(x'/a_h) - c(x/a_h),$$

which cannot be true given $a_h > 1$, $c''(x) \geq 0$ and $c'(x) > 0$.

Proof of Lemma 3

In a symmetric equilibrium, if an output is played with positive probability by one type of contestant, then it must be played with positive probability by both contestants of this type. Let $\hat{x} \in \{X_1^\ell \cup X_1^h\}$ be played with probability $q > 0$. Then

$$\Pr(\text{win}|\hat{x}) + \frac{q}{2} \leq \Pr(\text{win}|x) \text{ for all } x > \hat{x}.$$

Since for some ability, $\hat{x} \in BR(a)$, then $\pi(\hat{x}|a) \geq \pi(x|a)$ for all x. This implies that

$$\begin{aligned} &p_1 \Pr(\text{win}|\hat{x}) - c(\hat{x}/a^i) + \mathbb{E}[v_i(\mu(\hat{x}), \mu(x^{-i}), a^i)] \\ &\qquad \geq p_1 \Pr(\text{win}|x) - c(x/a^i) + \mathbb{E}[v_i(\mu(x), \mu(x^{-i}), a^i)] \end{aligned}$$

Combing the above inequalities,

$$p_1 \frac{q}{2} \leq \mathbb{E}[v_i(\mu(\hat{x}), \mu(x^{-i}), a^i)] - \mathbb{E}[v_i(\mu(x), \mu(x^{-i}), a^i)] + c(x/a^i) - c(\hat{x}/a^i).$$

By continuity of the cost function, $\exists \varepsilon > 0$ such that for all $x \in (\hat{x}, \hat{x}+\varepsilon)$, we have $c\left(\frac{\hat{x}+\varepsilon}{a^i}\right) - c\left(\frac{\hat{x}}{a^i}\right) < p_1\frac{q}{2}$. Then for each x in this range

$$\mathbb{E}[v_i(\mu(\hat{x}), \mu(x^{-i}), a^i)] - \mathbb{E}[v_i(\mu(x), \mu(x^{-i}), a^i)] > 0.$$

From Proposition 2, if $a^i = a_\ell$, then $\mu(\hat{x}) > \mu(x)$ and $\hat{x} \in \{X_1^\ell \cap X_1^h\}$. Similarly, if $a^i = a_h$, then $\mu(\hat{x}) < \mu(x)$ and $\hat{x} \in \{X_1^\ell \cap X_1^h\}$. In either case, $\hat{x} \in \{BR(a_\ell) \cap BR(a_h)\}$. However, the inequality cannot hold for both a_ℓ and a_h at the same time.

Note that we now can use the fact that $F_1(x)$ is continuous in x and we have that $\Pr(\text{win}|x) = \Pr(x < x^{-i}) = \Pr(x \leq x^{-i}) = F_1(x)$. Combined with Lemma 2, we have $\Pr(\mu(x^{-i}) < \mu(x)) \leq \Pr(\text{win}|x) = F_1(x) \leq \Pr(\mu(x^{-i}) \leq \mu(x))$.

Proof of Lemma 4

The proof follows in four steps.

(1) We first show that $x_{\ell*} = 0$. We do this by first showing that $x_{\ell*} \leq x_{h*}$, and then showing that $x_{\ell*}$ cannot be larger than zero.

Let $x_{h*} < x_{\ell*}$. Since $x_{h*} = \inf\{X_1^h\}$, $\forall \varepsilon > 0$, $\exists x_\varepsilon$ such that $x_{h*} \leq x_\varepsilon < x_{h*} + \varepsilon$ and $x_\varepsilon \in X_1^h$. In particular, this holds for $\varepsilon^* = x_{\ell*} - x_{h*}$. Then $x_{\varepsilon^*} \in \{X_1^h \setminus X_1^\ell\}$ and $\mu(x_{\varepsilon^*}) = 1$. However, from Lemma 2 we would have $\mu(x) = 1$ for all $x \in X_1^\ell$, which cannot hold. Therefore $x_{h*} \geq x_{\ell*}$.

If $0 < x_{\ell*} < x_{h*}$, then by Lemma 3, $\exists \delta$ with $0 < \delta < x_{h*} - x_{\ell*}$ such that $\forall x \in (x_{\ell*}, x_{\ell*}+\delta)$ we have $|p_1(F_1(x) - F_1(0))| = |p_1(F_1(x) - F_1(x_{\ell*}))| < c(x_{\ell*})$. Let $x_\delta \in X_1^\ell \cap (x_{\ell*}, x_{\ell*}+\delta)$. Then $\mu(x_\delta) = 0$ and $p_1(F_1(x_\delta) - F_1(0)) < c(x_\delta)$. Therefore,

$$\begin{aligned}\mathbb{E}[\pi^i(0)|a_\ell] &= p_1F_1(0) + \mathbb{E}[v_i(\mu(0), \mu(x^{-i}), a_\ell)] \\ &> p_1F_1(x_\delta) + \mathbb{E}[v_i(\mu(x_\delta), \mu(x^{-i}), a_\ell)] - c(x_\delta) \\ &= \mathbb{E}[\pi^i(x_\delta)|a_\ell].\end{aligned}$$

Then $x_\delta \notin BR(a_\ell)$, a contradiction.

If $0 < x_{\ell*} = x_{h*}$, then $\exists x_\ell, x_h$ such that $x_\ell \leq x_h$, $x_\ell \in X_1^\ell$, $x_h \in X_1^h$, and $p_1(F_1(x_\ell)-F_1(x_{\ell*})) = p_1F_1(x_\ell) < c(x_{\ell,*}) < c(x_\ell)$ and $p_1(F_1(x_h)-F_1(x_{h*})) = p_1F_1(x_h) < c(x_{h*}/a_h) < c(x_h/a_h)$, by the continuity of F_1.

$x_\ell \in X_1^\ell$ implies that

$$\begin{aligned}\mathbb{E}[\pi^i(x_\ell)|a_\ell] &= p_1F_1(x_\ell) - c(x_\ell) + E[v_i(\mu(x_\ell), \mu(x^{-i}), a_\ell)] \\ &\geq p_1F_1(0) - c(0) + \mathbb{E}[v_i(\mu(0), \mu(x^{-i}), a_\ell)] = \mathbb{E}[\pi^i(0)|a_\ell]\end{aligned}$$

This can hold only if $\mathbb{E}[v_i(\mu(x_\ell), \mu(x^{-i}), a_\ell)] > \mathbb{E}[v_i(\mu(0), \mu(x^{-i}), a_\ell)]$, which implies that $\mu(x_\ell) > \mu(0)$.

Similarly, $x_h \in X_1^h$ implies that $\mu(x_h) < \mu(0)$. Combining these two inequalities leads to $\mu(x_h) < \mu(x_\ell)$. This contradicts Lemma 2. Therefore we must have $0 = x_{\ell*} \leq x_{h*}$.

(2) We next show that $x_{h*} \leq x_\ell^*$.

If $x_\ell^* < x_{h*}$, then $\forall x \in (x_\ell^*, x_{h*})$, $x \notin \{X_1^\ell \cup X_1^h\}$. Let $x' = \frac{x_\ell^* + x_{h*}}{2}$ and $\varepsilon = c(x_{h*}/a_h) - c(x'/a_h)$. There is a $\delta > 0$ such that $\forall x \in (x_{h*}, x_{h*} + \delta)$, $p_1(F(x) - F(x_{h*})) < \varepsilon$. Pick an x_δ such that $x_\delta \in (x_{h*}, x_{h*} + \delta)$ and $x_\delta \in X_1^h$. Then $p_1(F_1(x_\delta) - F_1(x_{h*})) = p_1(F_1(x_\delta) - F_1(x')) < \varepsilon$, $c(x_\delta/a_h) - c(x'/a_h) > \varepsilon$, and $E[v_i(\mu(x_\delta), \mu(x^{-i}), a_h)] \leq E[v_i(\mu(x'), \mu(x^{-i}), a_h)]$. Therefore

$$\begin{aligned}\mathbb{E}[\pi^i(x')|a_h] &= p_1 F_1(x') + \mathbb{E}[v_i(\mu(x'), \mu(x^{-i}), a_h)] - c\left(\frac{x'}{a_h}\right) \\ &> p_1 F(x_\delta) + \mathbb{E}[v_i(\mu(x_\delta), \mu(x^{-i}), a_h)] - c\left(\frac{x_\delta}{a_h}\right) = \mathbb{E}[\pi^i(x_\delta)|a_h],\end{aligned}$$

a contradiction. So we can conclude that $x_\ell^* \leq x_{h*}$.

Also $x_\ell^* \leq x_h^*$. If we assume otherwise, then we can find $x \in \{X_1^\ell \setminus X_1^h\}$ where $x > x_h^*$ and $\mu(x) = 0$. Lemma 2 rules out this possibility.

We have shown so far that $0 = x_{\ell*} \leq x_{h*} \leq x_\ell^* \leq x_h^*$.

(3) For all $x \in (x_{\ell*}, x_{h*})$, $x \in BR(a_\ell)$ and for all $x \in (x_\ell^*, x_h^*)$, $x \in BR(a_h)$.

Given $x_{\ell*} < x_{h*}$, let $X_c^\ell = \{x | x \in \{(x_{\ell*}, x_{h*}) \setminus BR(a_\ell)\}\}$. If $x \in X_c^\ell$, then $\exists \varepsilon > 0$ such that $\mathbb{E}[\pi^i(x)|a_\ell] < \mathbb{E}[\pi^i(x')|a_\ell] - \varepsilon$ for all $x' \in \{(x_{\ell,*}, x_{h,*}) \cap X_1^\ell\}$. This implies that:

$$p_1 F_1(x) + \mathbb{E}[v_i(\mu(x), \mu(x^{-i}), a_\ell)] - c(x) < p_1 F_1(x') + \mathbb{E}[v_i(\mu(x'), \mu(x^{-i}), a_\ell)] - c(x') - \varepsilon,$$

where $\mathbb{E}[v_i(\mu(x), \mu(x^{-i}), a_\ell)] \geq \mathbb{E}[v_i(\mu(x'), \mu(x^{-i}), a_\ell)]$ as $\mu(x') = 0$. Therefore $p_1 F_1(x) - c(x) < p_1 F_1(x') - c(x') - \varepsilon$, and for all $x' > x$ with $x' \in \{(x_{\ell*}, x_{h*}) \cap X_1^\ell\}$, $p_1(F_1(x') - F_1(x)) > c(x') - c(x) - \varepsilon$.

Since F_1 and c are continuous, then there is a $\delta(\varepsilon) > 0$ such that for all $x' \in X_1^\ell$, $|x' - x| \geq \delta(\varepsilon)$. This implies that x is contained in an interval which is a subset of X_c^ℓ. Let a and b be the infimum and supremem of this interval respectively.

- If $b < x_{h*}$, then $\exists x' < x_{h*}, x' \in X_1^\ell$ where $|x' - b| < \delta, \forall \delta > 0$. Then, by the continuity of F_1, $\exists x' \in X_1^\ell$ and $p_1(F(x') - F(b)) < c(b) - c(\frac{a+b}{2})$. Then we know that

  $$\begin{gathered} p_1 F_1(x') - p_1 F_1\left(\frac{a+b}{2}\right) < c(b) - c\left(\frac{a+b}{2}\right) \text{ and} \\ \mathbb{E}[v_i(\mu(x'), \mu(x^{-i}), a_\ell)] \leq \mathbb{E}\left[v_i\left(\mu\left(\frac{a+b}{2}\right), \mu(x^{-i}), a_\ell\right)\right]. \end{gathered}$$

  This implies that $\mathbb{E}[\pi^i(x')|a_\ell] < \mathbb{E}[\pi^i(\frac{a+b}{2})|a_\ell]$ which contradicts $x' \in BR(a_\ell)$.

- If $b = x_{h*}$, then $\forall \delta > 0$, $\exists x' \in X_1^h$, s.t. $|x' - x_{h*}| < \delta$. We can take $x' \in X_1^h$ such that $p_1(F_1(x') - F_1(x_{h*})) < c(\frac{b}{a_h}) - c(\frac{a + x_{h*}}{2a_h})$.

  - If $x' \notin X_1^\ell$ then $\mu(x') = 1$, but since $\mathbb{E}[v_i(\mu(x'), \mu(x^{-i}), a_h)] \leq \mathbb{E}\left[v_i\left(\mu\left(\frac{a+x_{h*}}{2}\right), \mu(x^{-i}), a_h\right)\right]$, then this contradicts $x' \in BR(a_h)$.

– If $x' \in X_1^\ell$, then $\mu(x') \in [0,1]$. If $\mu(x') \leq \mu(\frac{a+x_{h*}}{2})$, then this contradicts $x' \in BR(a_\ell)$, but if $\mu(x') \geq \mu(\frac{a+x_{h*}}{2})$, this contradicts $x' \in BR(a_h)$.

Therefore X_c^ℓ must be empty.

Similarly, if $X_c^h = \{x | x \in \{(x_\ell^*, x_h^*) \setminus BR(a_h)\}$ and $x \in X_c^h$, then $\exists \varepsilon > 0$ such that $E[\pi(x)|a_h] < E[\pi(x')|a_h] - \varepsilon$ for all $x' \in X_1^h$ and therefore $|x' - x| \geq \delta(\varepsilon) > 0,\ \forall x' \in BR(a_h)$. Take a and b to be the infimum and supremum respectively of the interval of X_c^h containing x. Note that $b < x_h^*$, by the definition of x_h^*.

Now, there is an $x' \in X_1^h$ where $|x' - b| < \delta$ for all $\delta > 0$. Therefore there is an $x' \in BR(a_h)$ such that $p_1(F_1(x') - F_1(b)) < c(\frac{b}{a_h}) - c(\frac{b+a}{2a_h})$. Note that this implies that $p_1(F_1(x') - F_1(\frac{b+a}{2})) < c(\frac{x'}{a_h}) - c(\frac{b+a}{2a_h})$. However, this implies that

$$\begin{aligned}\mathbb{E}\left[\pi^i\left(\frac{b+a}{2}\right)|a_h\right] &= p_1F_1\left(\frac{b+a}{2}\right) - c\left(\frac{b+a}{2a_h}\right) + \mathbb{E}\left[v_i\left(\mu\left(\frac{b+a}{2}\right), \mu(x^{-i}), a_h\right)\right] \\ &> p_1F_1(x') - c\left(\frac{x'}{a_h}\right) + \mathbb{E}[v_i(\mu(x'), \mu(x^{-i}), a_h)] = \mathbb{E}[\pi^i(x')|a_h].\end{aligned}$$

This contradicts $x' \in BR(a_h)$, and therefore X_c^h must be empty.

(4) Lastly, we show that $x_{h*} < x_\ell^*$, and for all $x \in (x_{h*}, x_\ell^*)$, $x \in \{BR(a_\ell) \cap BR(a_h)\}$.

If $x_\ell^* = x_{h*}$, then $\forall \delta > 0$, there is $x_\ell \in X_1^\ell$ and $x_h \in X_1^h$ where $|x_h - x_\ell| < \delta$. By the continuity of F_1 and c, there is x_h and x_ℓ for which

$$\begin{aligned}p_1F_1(x_h) - c\left(\frac{x_h}{a_h}\right) &- \left(p_1F_1(x_\ell) - c\left(\frac{x_\ell}{a_h}\right)\right) \\ &< \mathbb{E}[v_i(\mu(x_\ell), \mu(x^{-i}), a_h)] - \mathbb{E}[v_i(\mu(x_h), \mu(x^{-i}), a_h)]\end{aligned}$$

since $\mu(x_\ell) = 0$, $\mu(x_h) = 1$, and $\mathbb{E}[v_i(0, \mu(x^{-i}), a_h)] - \mathbb{E}[v_i(1, \mu(x^{-i}), a_h)] > 0$. Then

$$\begin{aligned}\mathbb{E}[\pi^i(x_\ell)|a_h] &= p_1F_1(x_\ell) - c\left(\frac{x_\ell}{a_h}\right) + \mathbb{E}[v_i(\mu(x_\ell), \mu(x^{-i}), a_h)] \\ &> p_1F_1(x_h) - c\left(\frac{x_h}{a_h}\right) + \mathbb{E}[v_i(\mu(x_h), \mu(x^{-i}), a_h)] = \mathbb{E}[\pi^i(x_h)|a_h],\end{aligned}$$

which cannot be true as $x_h \in BR(a_h)$.

Define $X_c = \{x | x \in (x_{h*}, x_\ell^*) \setminus (BR(a_\ell) \cup BR(a_h))\}$. From Lemma 2, we know that for all $x' \in \{(x_{h*}, x_\ell^*) \cap (X_1^\ell \cup X_1^h)\}$, $\mu(x') \in (0,1)$ as $\mu(x') = 1$, implies $x_\ell^* \leq x'$ and $\mu(x') = 0$ implies $x_{h*} \geq x'$. Therefore $x' \in \{X_1^\ell \cap X_1^h\}$.

Let $x \in X_c$ be given. Then for all $x', x'' \in \{(x_{h*}, x_\ell^*) \cap (X_1^\ell \cap X_1^h)\}$ such that $x' < x < x''$ we must have $\mu(x') \leq \mu(x'')$. Let $\mu^* \in [\sup\{\mu(x')\}, \inf\{\mu(x'')\}]$. These are well-defined as there is at least one such x' and x''.

If $\mu(x) \geq \mu^*$ then $\mathbb{E}[v_i(\mu(x), \mu(x^{-i}), a_\ell)] \geq \mathbb{E}[v_i(\mu(x'), \mu(x^{-i}), a_\ell)]$ for all x' as defined above. Therefore

$$p_1 F_1(x') - c(x') + \mathbb{E}[v_i(\mu(x'), \mu(x^{-i}), a_\ell)] - \varepsilon_1 > p_1 F_1(x) - c(x) + \mathbb{E}[v_i(\mu(x), \mu(x^{-i}), a_\ell)]$$
$$\Rightarrow p_1 F_1(x') - c(x') - \varepsilon_1 > p_1 F_1(x) - c(x)$$

By continuity of F_1 and c, $\exists \delta_1 > 0$ such that $\forall x'$, $|x' - x| > \delta_1$. Then $[x - \delta_1, x] \subset X_c$.

Similarly, if $\mu(x) < \mu^*$, then $\mathbb{E}[v_i(\mu(x), \mu(x^{-i}), a_h)] \geq \mathbb{E}[v_i(\mu(x''), \mu(x^{-i}), a_h)]$ for all x'' and

$$\Rightarrow p_1 F_1(x'') - c\left(\frac{x''}{a_h}\right) - \varepsilon_2 > p_1 F_1(x) - c\left(\frac{x}{a_h}\right)$$

Then, by continuity, $\exists \delta_2 > 0$ such that $\forall x''$, $|x'' - x| > \delta_2$. Then $[x, x + \delta_2] \subset X_c$. In either case, if $x \in X_c$, then there is an interval with some supremum b and infimum a such that $x \in (a, b) \subset X_c$.

If $b < x_\ell^*$, then there is an $x' \in \{(x_{h*}, x_\ell^*) \cap X_1^\ell \cap X_1^h\}$ where $|x' - b| < \delta$ for all $\delta > 0$. Therefore there is an $x' \in \{(x_{h*}, x_\ell^*) \cap X_1^\ell \cap X_1^h\}$ such that $p_1(F(x') - F(b)) < c(b/a_h) - c(\frac{b+a}{2a_h})$. This implies that $p_1(F_1(x') - F_1(\frac{b+a}{2})) < c(x'/a_h) - c(\frac{b+a}{2a_h})$ and $p_1(F_1(x') - F_1(\frac{b+a}{2})) < c(x') - c(\frac{b+a}{2})$.

If $\mu((b + a)/2) < \mu(x')$ then

$$\mathbb{E}\left[\pi^i\left(\frac{b+a}{2}\right)|a_h\right] = p_1 F_1\left(\frac{b+a}{2}\right) - c\left(\frac{b+a}{2a_h}\right) + \mathbb{E}\left[v_i\left(\mu\left(\frac{b+a}{2}\right), \mu(x^{-i}), a_h\right)\right]$$
$$> p_1 F_1(x') - c\left(\frac{x'}{a_h}\right) + \mathbb{E}[v_i(\mu(x'), \mu(x^{-i}), a_h)] = \mathbb{E}[\pi^i(x')|a_h].$$

If $\mu((b + a)/2) \geq \mu(x')$ then

$$\mathbb{E}\left[\pi^i\left(\frac{b+a}{2}\right)|a_\ell\right] = p_1 F_1\left(\frac{b+a}{2}\right) - c\left(\frac{b+a}{2}\right) + \mathbb{E}\left[v_i\left(\mu\left(\frac{b+a}{2}\right), \mu(x^{-i}), a_\ell\right)\right]$$
$$> p_1 F_1(x') - c(x') + \mathbb{E}[v_i(\mu(x'), \mu(x^{-i}), a_\ell)] = \mathbb{E}[\pi^i(x')|a_\ell].$$

In either case, this contradicts $x' \in \{X_1^\ell \cap X_1^h\}$.

If $b = x_\ell^*$, then there is an $x' \in X_1^h$, such that $|x' - b| < \delta$, and $\mu(x') = 1$. This implies that $p_1(F_1(x') - F_1(\frac{b+a}{2})) < c(x'/a_h) - c(\frac{b+a}{2a_h})$, and

$$\mathbb{E}\left[\pi^i\left(\frac{b+a}{2}\right)|a_h\right] = p_1 F_1\left(\frac{b+a}{2}\right) - c\left(\frac{b+a}{2a_h}\right) + \mathbb{E}\left[v_i\left(\mu\left(\frac{b+a}{2}\right), \mu(x^{-i}), a_h\right)\right]$$
$$> p_1 F_1(x') - c\left(\frac{x'}{a_h}\right) + \mathbb{E}[v_i(\mu(x'), \mu(x^{-i}), a_h)] = \mathbb{E}[\pi^i(x')|a_h].$$

This contradicts $x' \in X_1^h$. Therefore X_c must be empty and for all $x \in (x_{h*}, x_\ell^*)$, we must have $x \in \{BR(a_\ell) \cap BR(a_h)\}$.

Proof of Lemma 5

Lemma 3 shows that no output is played with positive probability by either low or high ability contestants. Therefore H_1 and L_1 are continuous and $F_1 = (1-\hat{\mu})L_1 + \hat{\mu}H_1$ is also continuous.

To show that $\mu(x)$ is continuous on $(0, x_h^*)$, note that $\mathbb{E}[\pi^i(x)|a_\ell]$ is constant for all $x \in BR(a_\ell)$ and $\mathbb{E}[\pi^i(x)|a_h]$ is constant for all $x \in BR(a_h)$. Since both $F_1(x)$ and $c(x)$ are continuous on $(0,\infty)$ and $\mathbb{E}[v_i(\mu(x), \mu(x^{-i}), a_\ell)] = c(x,1) - p_1F_1(x) + K_\ell(p_1,p_2)$ on $[0, x_\ell^*]$, then $\mathbb{E}[v_i(\mu(x), \mu(x^{-i}), a_\ell)]$ must be continuous on this interval. Similarly, $\mathbb{E}[v_i(\mu(x), \mu(x^{-i}), a_h)]$ is continuous on this interval. Since $\mathbb{E}[v_i(\mu(x), \mu(x^{-i}), a_h)]$ is strictly decreasing in $\mu(x)$, and $\mathbb{E}[v_i(\mu(x), \mu(x^{-i}), a_\ell)]$ is strictly increasing in $\mu(x)$, then $\mu(x)$ must also be continuous on $BR(a_\ell) \cup BR(a_h) = [0, x_h^*]$.

We now show that the set $[0, x_h^*] \setminus X_1$ has no interior, i.e. there can be no interval $[a, b] \subset [0, x_h^*]$ where for all $x \in [a, b]$, $x \notin X_1$. This implies that X_1 is dense in $[0, x_h^*]$.

If we let $[\tilde{a}, \tilde{b}] \subset [0, x_h^*] \setminus X_1$ be given, then define a and b to be the infimum and supremum respectively of the interval in $[0, x_h^*] \setminus X_1$ which contains $[\tilde{a}, \tilde{b}]$. Neither x_{h*} nor x_ℓ^* can be contained in the interval as they are the limit point of a subset of X_1. Then the interval $[a, b]$ must be contained within either $[0, x_{h*}]$, $[x_{h*}, x_\ell^*]$, or $[x_\ell^*, x_h^*]$.

1. If $[a, b] \subset [0, x_{h*}]$, then for all $x \in [a, b]$, $x \in BR(a_\ell)$ and $F_1(x) = F_1(a)$. Therefore, $\mathbb{E}[v_i(\mu(b), \mu(x^{-i}), a_\ell)] - c(b) = \mathbb{E}[v_i(\mu(a), \mu(x^{-i}), a_\ell)] - c(a)$, which implies that $\mu(b) > \mu(a)$. Since $\mu(x)$ is continuous, then for all $\delta > 0$, there is an $x \in X_1^h$ such that $|x - b| < \delta$ and $\mu(x) > 0$. If $x \in X_1^h \setminus X_1^\ell$, then $\mu(x)$=1, and $\mathbb{E}[\pi^i(\frac{a+b}{2})|a_h] > \mathbb{E}[\pi^i(x)|a_h]$, a contradiction. If $x \in X_1^h \cap X_1^\ell$ then depending on the value of $\mu((a+b)/2)$, either $\mathbb{E}[\pi^i(\frac{a+b}{2})|a_\ell] > \mathbb{E}[\pi^i(a)|a_\ell]$ or $\mathbb{E}[\pi^i(\frac{a+b}{2})|a_h] > \mathbb{E}[\pi^i(x)|a_h]$, again a contradiction.

2. If $[a, b] \subset [x_{h*}, x_\ell^*]$, then for all $x \in [a, b]$, $x \in \{BR(a_\ell) \cap BR(a_h)\}$ which implies

$$\mathbb{E}[v_i(\mu(b), \mu(x^{-i}), a_\ell)] - c(b) = \mathbb{E}[v_i(\mu(a), \mu(x^{-i}), a_\ell)] - c(a),$$
$$\mathbb{E}[v_i(\mu(b), \mu(x^{-i}), a_h)] - c(b/a_h) = \mathbb{E}[v_i(\mu(a), \mu(x^{-i}), a_h)] - c(a/a_h).$$

   However, rearranging these equations, it is clear they cannot hold at the same time as the right hand sides are both strictly positive which contradicts Proposition 2.

$$\mathbb{E}[v_i(\mu(b), \mu(x^{-i}), a_\ell)] - \mathbb{E}[v_i(\mu(a), \mu(x^{-i}), a_\ell)] = c(b) - c(a)$$
$$\mathbb{E}[v_i(\mu(b), \mu(x^{-i}), a_h)] - \mathbb{E}[v_i(\mu(a), \mu(x^{-i}), a_h)] = c(b/a_h) - c(a/a_h)$$

3. If $[a, b] \subset [x_\ell^*, x_h^*]$, then for all $x \in [a, b]$, $x \in BR(a_h)$ and therefore,

$$\mathbb{E}[v_i(\mu(b), \mu(x^{-i}), a_h)] - c(b/a_h) = \mathbb{E}[v_i(\mu(a), \mu(x^{-i}), a_h)] - c(a/a_h),$$

   and $\mu(b) < \mu(a) \leq 1$. Then for all $\delta > 0$, there is an $x \in X_1^h$ such that $|x - b| < \delta$ and $\mu(x) = 1$. However, this contradicts the continuity of $\mu(x)$.

If $\mu(x) = \varepsilon > 0$, then by the continuity of $\mu(x)$, $\exists \delta > 0$ where $\forall x', |x' - x| < \delta$,

$\mu(x) > \varepsilon/2$. However for all $\delta > 0$ there is an $x' \in X_1^\ell \setminus X_1^h$ for which $\mu(x') = 0$, a contradiction. Therefore $\mu(x) = 0$ for all $x \in [0, x_{h*})$. Note that $\mu(x_{h*}) = 0$ when $x_{h*} > 0$, which follows from continuity from the left. Additionally, $\mu(x) = 1$ for all $x \in [x_\ell^*, x_h^*]$ when $x_\ell^* < x_h^*$. To show that $\mu(x)$ is weakly increasing on $[x_{h*}, x_\ell^*]$, let $x, y \in [x_{h*}, x_\ell^*]$ be such that, $\mu(x) > \mu(y)$ and $x < y$. Then there is an x' and y' arbitrarily close to x and y respectively, where $x', y' \in X_1$ and therefore $\mu(x') \leq \mu(y')$. This is not consistent with $\mu(\cdot)$ being continuous, a contradiction.

Proof of Theorem 1

There are up to three distinct intervals in each equilibrium. We will show that the end-points of these intervals and the distribution functions on the intervals are completely determined by the first order conditions of the contestants.

Conditions for x being in $BR(a_h)$ and $BR(a_\ell)$ are[14]

$$BR(a_h): \; p_1 F_1^*(x) + \mathbb{E}[v_i(\mu(x), \mu(x^{-i}), a_h)] - c\left(\frac{x}{a_h}\right) = K_h(p_1, p_2),$$
$$BR(a_\ell): \; p_1 F_1^*(x) + \mathbb{E}[v_i(\mu(x), \mu(x^{-i}), a_\ell)] - c(x) = K_\ell(p_1, p_2) = 0.$$

For all values of p_1 and p_2, Lemma 4 shows that $x_{h*} < x_\ell^*$, and therefore the interval $[x_{h*}, x_\ell^*]$ is non-trivial. On this interval, $x \in \{X_1^\ell \cup X_1^h\}$ implies $x \in \{X_1^\ell \cap X_1^h\}$. Subtracting the condition for X_1^ℓ from the condition for X_1^h

$$\mathbb{E}[v_i(\mu(x), \mu(x^{-i}), a_h)] - \mathbb{E}[v_i(\mu(x), \mu(x^{-i}), a_\ell)] = c\left(\frac{x}{a_h}\right) - c(x) + K_h(p_1, p_2).$$

Taking the derivative of each side with respect to output gives Equation (1).

$$\mu'(x) d(\mu(x)) = c'(x) - \frac{1}{a_h} c'\left(\frac{x}{a_h}\right)$$

Note that on this interval, $\mu'(x) > 0$ and therefore, $F_\mu(\mu(x)) = F_1^*(x)$.

Taking the derivative of the condition for X_1^ℓ and combining with Equation (1) we get Equation (2).

$$p_1 f_1^*(x) + \mu'(x) d(\mu(x)) F_1^*(x) = c'(x)$$
$$\Rightarrow p_1 f_1^*(x) = c'(x)(1 - F_1^*(x)) + \frac{1}{a_h} c'\left(\frac{x}{a_h}\right) F_1^*(x).$$

From continuity of $F_1^*(x)$, $p_1 F_1^*(x_{h*}) = c(x_{h*})$. For a given x_{h*}, using the Picard - Lindelöf Theorem[15], we know that there is a unique solution for $f_1^*(x)$ on $[x_{h*}, x_\ell^*]$, and therefore $F_1^*(x)$ is determined on this interval.

[14]The constants $K_h(p_1, p_2)$ and $K_\ell(p_1, p_2)$ are the two contest expected payoffs of the high and low ability contestants for given prizes (p_1, p_2). The expected payoff of the low ability contestant is zero, as choosing $x = 0$ in the first contest is always in the best response set. This output guarantees the contestant zero payoff in the first contest and the weak position entering the second contest leading to a zero expected payoff in the second contest.

[15]The right hand side of equation (2) is continuous in x and uniformly Lipshitz continuous in $F_1^*(x)$ on the interval of $[x_{h*}, x_\ell^*]$. Also, due to the properties of the cost function, the distribution function is bounded between 0 and 1.

To see why only one such x_{h*} can lead to an equilibrium, consider a different initial condition, $p_1 \tilde{F}^*(\tilde{x}_{h*}) = c(\tilde{x}_{h*})$ where $\tilde{x}_{h*} > x_{h*}$ and the associated $\tilde{f}_1(x)$ on $[\tilde{x}_{h*}, \tilde{x}^*_\ell]$. Then both $\tilde{F}^*_1(\tilde{x}_{h*}) > F^*_1(\tilde{x}_{h*})$ and $\tilde{\mu}(\tilde{x}_{h*}) < \mu(\tilde{x}_{h*})$, and for all $x \in [\tilde{x}_{h*}, x^*_\ell]$, $\tilde{F}^*_1(x) > F^*_1(x)$, $\tilde{f}^*_1(x) < f^*_1(x)$, and $\mu(x) > \tilde{\mu}(x)$. This implies that $\tilde{H}(x^*_\ell) = \int_0^{x^*_\ell} \tilde{\mu}(x)\tilde{f}(x)dx < \int_0^{x^*_\ell} \mu(x)f(x)dx = H(x^*_\ell)$ and therefore $\tilde{L}(x^*_\ell) > L(x^*_\ell) = 1$, a contradiction. Similarly, there cannot be an additional equilibrium where $\tilde{x}_{h*} < x_{h*}$.

The belief function on this interval is determined up to a constant by equation (1). The constant is determined by $\mu(x_{h,*})$ which is 0 when $x_{h*} > 0$, and needs to be characterized in equilibrium when $x_{h*} = 0$. Given this constant, the equilibrium strategies of high ability and low ability contestants can be constructed on this interval.

For small values of p_1 relative to p_2, this is the only non-trivial interval: $x_{h*} = 0$ and $x^*_\ell = x^*_h$. In this case, $\mu(x_{h*}) \in [0, 1/2]$ and $\mu(x^*_h) \in [1/2, 1]$ both need to be determined in equilibrium along with x^*_h. By an argument similar to that for showing x_{h*} is unique, if $x_{h*} = 0$ then $\mu(x_{h*})$ is also uniquely determined. Then $\mu(x)$ and $F(x)$ are uniquely determined on this interval, and therefore x^*_h and $\mu(x^*_h)$ are also uniquely determined.

For larger p_1, $x_{h*} > 0$ and/or $x^*_h > x^*_\ell$. When the intervals are non-trivial, then the belief functions on these intervals were characterized in Lemma 5. Characterization of the output distributions directly follow. For $x \in [0, x_{h*})$, $\mathbb{E}[v_i(\mu(x), \mu(x^{-i}), a_\ell)] = 0$ as $\mu(x) = 0$,and therefore $p_1 F^*_1(x) = c(x)$. For all $x \in [x^*_\ell, x^*_h]$, $\mathbb{E}[v_i(\mu(x), \mu(x^{-i}), a_h)] = \mathbb{E}[v_i(1, \mu(x^{-i}), a_h)] \equiv v_h(p_2)$ and $p_1 F^*_1(x) + v_h(p_2) = c(x/a_h) + K_h(p_1, p_2)$, .

Given $F_1(x)$ and $\mu(x)$ on $[0, x^*]$, the output distribution of both the low and high ability contestants can be determined. Therefore $F^*_1(x), L^*_1(x)$ and $H^*_1(x)$ are uniquely characterized on X_1 where $\overline{X}_1 = [0, x^*_h]$. These distributions along with the second period output distributions $L^*_2(x|\mu_i, \mu_{-i})$ and $H^*_2(x|\mu_i, \mu_{-i})$ form the unique symmetric Bayes Nash equilibrium.

Proof of Proposition 3

The best response set of the low ability contestant always contains zero. The payoff in the first contest for this output choice is zero. Because the belief function is weakly increasing in output, choosing zero will ensure that the contestant enters the second contest in the weak (or an identical) position. In these cases, the expected payoff of a low ability contestant in the second contest is zero.

The output, $x_{h,*}$, is in the best response set of the high ability player. Then two period payoffs are

$$\begin{aligned} K_h(p_1, p_2) &= p_1 F^*_1(x_{h,*}) - c\left(\frac{x_{h,*}}{a_h}\right) + \mathbb{E}[v_w(\mu(x_{h,*}), \mu_{-i}, a_h)] \\ &= c(x_{h,*}) - c\left(\frac{x_{h,*}}{a_h}\right) + p_2(1 - \mu(x_{h,*})) - c\left(\frac{c^{-1}(p_2(1 - \mu(x_{h,*})))}{a_h}\right). \end{aligned}$$

Given the assumptions on the cost function, this gives the desired expression.

Proof of Proposition 4

Belief distributions that arise after the first contest for different prize structures must be equal at least at one point. If the distributions do not cross then one distribution FOSD the other and the distributions cannot have the same expected value. However, the expectation of the probability that a contestant is high ability is 1/2 in either case.

Let $\tilde{\mu}(\hat{\tilde{x}}) = \mu(\hat{x}) = \hat{M}$ be a point of intersection for belief distributions $\tilde{F}_\mu(M)$ and $F_\mu(M)$. Note that

$$f_\mu(\hat{M}) = \frac{\partial}{\partial \mu} F_1(\mu^{-1}(\hat{M})) = \frac{f_1(\mu^{-1}(\hat{M}))}{\mu'(\mu^{-1}(\hat{M}))} = \frac{f_1(\hat{x})}{\mu'(\hat{x})}.$$

From equations (1) and (2),

$$\begin{aligned} \frac{f_1(\hat{x})}{\mu'(\hat{x})} &= \frac{d(\mu(\hat{x}))\left(c'(x) - F_1(\hat{x})\left(c'(x) - \frac{1}{a_h}c'\left(\frac{\hat{x}}{a_h}\right)\right)\right)}{p_1\left(c'(x) - \frac{1}{a_h}c'\left(\frac{\hat{x}}{a_h}\right)\right)} = \frac{d(\mu(\hat{x}))}{p_1}\left(\frac{a_h^\alpha}{a_h^\alpha - 1} - F_1(\hat{x})\right) \\ &= \frac{p_2}{p_1}\left(\frac{a_h^\alpha(1 - F_1(\hat{x})) + F_1(\hat{x})}{a_h^\alpha}\right) \end{aligned}$$

Because $\tilde{F}_1(\tilde{\mu}^{-1}(\hat{M})) = F_1(\mu^{-1}(\hat{M}))$, then $\tilde{f}_\mu(\hat{M}) \lesseqgtr f_\mu(\hat{M})$ when $\frac{\tilde{p}_2}{\tilde{p}_1} \lesseqgtr \frac{p_2}{p_1}$. Given that $\tilde{f}_\mu(\hat{M}) < f_\mu(\hat{M})$, as in the case when first contest prize is increased for a fixed second contest prize, this implies that $\tilde{F}_\mu(M)$ crosses $F_\mu(M)$ exactly once from above and $\tilde{F}_\mu(M) <_{SOSD} F_\mu(M)$. An increase the second contest prize for fixed first contest prize implies $\tilde{F}_\mu(M)$ crosses $F_\mu(M)$ exactly once from below and $\tilde{F}_\mu(M) >_{SOSD} F_\mu(M)$.

Proof of Corollary 2

The probability that the low ability player wins the first contest is given by

$$\begin{aligned} \int_0^1 \ell_\mu(M)F_\mu(M)dM &= \int_0^1 2(1-M)f_\mu(M)F_\mu(M)dM \\ &= 2\int_0^1 f_\mu(M)F_\mu(M)dM - 2\int_0^1 mf_\mu(M)F_\mu(M)dM \\ &= \int_0^1 (F_\mu(M))^2 dM - (F_\mu(0))^2, \end{aligned}$$

where the last equality follows from integration by parts.

Whenever $\tilde{F}_\mu >_{SOSD} F_\mu$ then $F_\mu(0) \geq \tilde{F}_\mu(0)$ and $\int_0^1 (F_\mu(M))^2 dM < \int_0^1 (\tilde{F}_\mu(M))^2 dM$.

The later follows from

$$\begin{aligned}\int_0^1 (\tilde{F}_\mu(M))^2 - (F_\mu(M))^2 dM &= \int_0^1 (\tilde{F}_\mu(M) + F_\mu(M))(\tilde{F}_\mu(M) - F_\mu(M))dM \\ &> (\tilde{F}_\mu(\hat{M}) + F_\mu(\hat{M})) \int_0^1 \tilde{F}_\mu(M) - F_\mu(M) dM \\ &= (\tilde{F}_\mu(\hat{M}) + F_\mu(\hat{M}))(\tilde{F}_\mu(1) - F_\mu(1)) \geq 0.\end{aligned}$$

The middle inequality uses the fact that $\tilde{F}_\mu(\hat{M}) = F_\mu(\hat{M})$ with $\tilde{F}_\mu(M) \lesseqgtr F_\mu(M)$ when $M \lesseqgtr \hat{M}$. The last equality follows from $\int_0^1 F_\mu(M)dM = F_\mu(1) - \mathbb{E}[M]$, and the final inequality follows from $\tilde{F}_\mu(1) \geq F_\mu(1)$.

The probability that a low ability player wins against another low probability player is 1/2 in any symmetric equilibrium. Therefore the above inequalities imply that a higher price ratio, p_1/p_2, reduces the likelihood a low ability player wins against a high ability player.

The contestant that wins the first contest will enter the second contest in the stronger position. In equilibrium, the expected belief of the stronger contestant equals to the probability that the winner of the first contest is high ability. A higher price ratio, p_1/p_2 will increase this probability and therefore the expected belief of the stronger contestant. Similarly, the contestant that lost the first contest will be weaker in the second contest and a higher price ratio will decrease the expected probability that this contestant is high ability.

B Equilibrium construction

Assume the cost function takes the form, $c(x) = kx^\alpha$, with $\alpha \geq 1$ and $k > 0$ and let $\hat{\mu} = 1/2$. For the equilibrium of the first contest we find the ex-ante expected distribution of each contestant over each of the potential three ranges of output which depend on the values of p_1 and p_2.

For any values of p_1 and p_2, $x_{h*} < x_\ell^*$. For $x \in [x_{h*}, x_\ell^*]$ the expected output distribution satisfies equation (2). The family of solutions is

$$F_1^*(x) = Be^{(c(x/a_h) - c(x))/p_1} + \frac{a_h^\alpha}{a_h^\alpha - 1},$$

with boundary condition $F_1(x_{h*}) = \frac{1}{p_1} k x_{h*}^\alpha$. The solution is

$$F_1^*(x) = \frac{a_h^\alpha}{a_h^\alpha - 1} - \left(\frac{a_h^\alpha}{a_h^\alpha - 1} - \frac{1}{p_1} k x_{h*}^\alpha\right) e^{-\frac{a_h^\alpha - 1}{a_h^\alpha} \frac{1}{p_1}(kx^\alpha - kx_{h*}^\alpha)}.$$

The belief function satisfies the condition in equation (1) which simplifies under this parameterization to $p_2 \mu'(x) = c'(x)$. The belief function is $\mu(x) = \frac{1}{p_2}(kx^\alpha + C)$, where

$C = -c(x_{h*})$ if $x_{h*} > 0$ and $C = p_2\mu(x_{h*})$ if $x_{h*} = 0$. Therefore

$$F_1^*(x) = \frac{a_h^\alpha}{a_h^\alpha - 1} - \left(\frac{a_h^\alpha}{a_h^\alpha - 1} - \frac{1}{p_1}kx_{h*}^\alpha\right) e^{-\frac{(a_h^\alpha - 1)}{a_h^\alpha}\frac{p_2}{p_1}(\mu(x) - \mu(x_{h*}))},$$

where $\mu(x_{h*}) = 0$ when $x_{h*} > 0$.

If $x_{h*} > 0$, then $F_1^*(x) = \frac{1}{p_1}kx^\alpha$ and $\mu(x) = 0$ for $x \in [0, x_{h*}]$. If $x_\ell^* < x_h^*$, then $F_1^*(x) = \frac{1}{p_1}\left(\frac{1}{a_h^\alpha}kx^\alpha + K_h(p_1,p_2) - v_h(p_2)\right)$ and $\mu(x) = 1$ for $x \in [x_\ell^*, x_h^*]$.

Given $F_1^*(x)$ and $\mu(x)$, the output distribution of the both the high and low ability contestants comes from using $2F_1^*(x) = L_1^*(x) + H_1^*(x)$ and $\mu(x) = \frac{h_1^*(x)}{\ell_1^*(x) + h_1^*(x)}$.

Let $A = \frac{a_h^\alpha}{a_h^\alpha - 1}$. Over the range $x \in [x_{h,*}, x_\ell^*]$ these distributions are

$$H_1^*(x) = 2(A - \frac{1}{p_1}kx_{h,*}^\alpha)\left(\frac{Ap_1}{p_2} - (\mu(x) + \frac{Ap_1}{p_2})e^{-\frac{p_2}{Ap_1}(\mu(x) - \mu(x_{h,*}))}\right) + 2A\mu(x_{h,*})$$

$$L_1^*(x) = 2A(1 - \mu(x_{h,*})) + 2(A - \frac{1}{p_1}kx_{h,*}^\alpha)\left((\mu(x) + \frac{Ap_1}{p_2} - 1)e^{-\frac{p_2}{Ap_1}(\mu(x) - \mu(x_{h,*}))} - \frac{Ap_1}{p_2}\right)$$

Small prize in first contest

Given p_2, for p_1 close enough to 0 (specifically for $p_1 < \frac{p_2}{2}((A^2 - A)\log(a_h^\alpha) - A)^{-1}$), both $x_{h*} = 0$ and $x_\ell^* = x_h^*$. The expected output distribution becomes

$$F_1(x) = A\left(1 - e^{-\frac{p_2}{Ap_1}(\mu(x) - \mu(x_{h*}))}\right), \text{ for } 0 \le x \le x_h^*.$$

Given $H(x_{h*}) = F(x_{h*}) = F(0) = 0$, the output distribution of the high ability contestant is

$$H_1(x) = \int_0^x \mu(t)f_1(t)dt = 2F_1(x)\left(\mu(x) + \frac{Ap_1}{p_2}\right) - 2A(\mu(x) - \mu(x_{h*})).$$

Combining $F_1(x_h^*) = 1$ and $H_1(x_h^*) = 1$ gives

$$\mu(x_h^*) - \mu(x_{h*}) = \frac{p_1}{p_2} + \frac{2\mu(x_h^*) - 1}{2A}.$$

Plugging back into $F(x_h^*) = 1$, we can solve the belief function at each end point

$$\mu(x_h^*) = \frac{1}{2} + \frac{p_1}{p_2}(A^2\log(a_h^\alpha) - A) \text{ and } \mu(x_{h*}) = \frac{1}{2} + \frac{p_1}{p_2}((A^2 - A)\log(a_h^\alpha) - A)$$

Therefore $\mu(x_h^*) - \mu(x_{h*}) = \frac{p_1}{p_2}A\log(a_h^\alpha)$ and $kx_h^{*\alpha} = p_1A\log(a_h^\alpha)$.

Example of parameters that fall in this category: $c(e) = e^2$, $a_h = 2$ and $p_1 = .5$ and $p_2 = 1$.

Intermediate prize in first contest

For larger p_1 compared to p_2, (specifically $p_1 > \frac{p_2}{2}((A^2 - A)\log(a_h^\alpha) - A)^{-1}$), then $H(x_\ell^*) < 1$ and $x_\ell^* < x_h^*$. The expected output distribution is

$$F_1(x) = \begin{cases} A\left(1 - e^{\frac{-p_2}{Ap_1}(\mu(x)-\mu(x_{h*}))}\right) & 0 \leq x \leq x_\ell^* \\ \frac{1}{p_1}(\frac{k}{a_h^\alpha}x^\alpha + K_h(p_1,p_2) - v_h(p_2)) & x_\ell^* \leq x \leq x_h^* \end{cases}$$

The output distributions of the high and low ability contestants are

$$H_1(x) = 2F_1(x)\left(\mu(x) + \frac{Ap_1}{p_2}\right) - 2A(\mu(x) - \mu(x_{h*})).$$

$$L_1(x) = 2F_1(x)\left(1 - \mu(x) - \frac{Ap_1}{p_2}\right) + 2A(\mu(x) - \mu(x_{h*}))$$

To characterize the equilibrium we need to solve for μ_0, x_ℓ^*, x_h^*, and $K_h(p_1,p_2) - v_h(p_2)$.

1. Continuity of the belief function: $\mu(x_\ell^*) = 1$ implies that $kx_\ell^{*\alpha} = p_2(1 - \mu(x_{h*}))$.
2. $L_1^*(x_\ell^*) = 1$ gives an implicit equation for $\mu(x_{h*})$

$$1 = 2A\left((1 - \mu(x_{h*})) - \frac{Ap_1}{p_2}\left(1 - e^{-\frac{p_2}{Ap_1}(1-\mu(x_{h*}))}\right)\right)$$

3. By continuity of $F_1(x)$ at x_ℓ^*, we can find $K_h(p_1,p_2) - v_h(p_2)$.

$$A - Ae^{-\frac{p_2}{Ap_1}(1-\mu(x_{h*}))} = \frac{1}{p_1}(\frac{k}{a_h^\alpha}x_\ell^{*\alpha} + K_h(p_1,p_2) - v_h(p_2))$$

   Using the equation that determines $\mu(x_{h*})$ and the belief equations, this simplifies to

$$K_h(p_1,p_2) - v_h(p_2) = \frac{p_2}{A}\left(\frac{1}{2} - \mu(x_{h*})\right)$$

4. From $F(x_h^*) = 1$ we can find x_h^*.

$$kx_h^{*\alpha} = p_1 a_h^\alpha - p_2(a_h^\alpha - 1)(1/2 - \mu(x_{h*}))$$

Example of parameters that fall in this category: $c(e) = e^2$, $a_h = 2$ and $p_1 = 0.8$ and $p_2 = 1$.

Large prize in first contest

For large enough p_1, all three intervals are non-trivial, $\mu(x_{h*}) = 0$ and $\mu(x_\ell^*) = 1$. The distribution functions are

$$F_1^*(x) = \begin{cases} \frac{1}{p_1} k x^\alpha & 0 \le x \le x_{h*} \\ A - (A - \frac{1}{p_1} k x_{h*}^\alpha) e^{\frac{-p_2}{Ap_1}\mu(x)} & x_{h*} \le x \le x_\ell^* \\ \frac{1}{p_1}(\frac{k}{a_h^\alpha} x^\alpha + K_h(p_1,p_2) - v_h(p_2)) & x_\ell^* \le x \le x_h^* \end{cases}$$

$$L_1^*(x) = \begin{cases} \frac{2}{p_1} k x^\alpha, & 0 \le x \le x_{h*} \\ 2A + 2(A - \frac{1}{p_1} k x_{h*}^\alpha) \left((\mu(x) + \frac{Ap_1}{p_2} - 1) e^{-\frac{p_2}{Ap_1}\mu(x)} - \frac{Ap_1}{p_2} \right), & x_{h*} \le x \le x_\ell^* \\ 1, & x_\ell^* \le x \le x_h^* \end{cases}$$

$$H_1^*(x) = \begin{cases} 0, & 0 \le x \le x_{h*} \\ 2(A - \frac{1}{p_1} k x_{h*}^\alpha) \left(\frac{Ap_1}{p_2} - (\mu(x) + \frac{Ap_1}{p_2}) e^{-\frac{p_2}{Ap_1}\mu(x)} \right), & x_{h*} \le x \le x_\ell^* \\ \frac{2}{p_1}(\frac{k}{a_h^\alpha} x^\alpha + K_h(p_1,p_2) - v_h(p_2)) - 1 & x_\ell^* \le x \le x_h^* \end{cases}$$

Using $\mu(x_\ell^*) = 1$ and $L(x_\ell^*) = 1$ identifies the endpoints of the middle interval.

$$k x_{h*}^\alpha = p_1 \left(A - \frac{(2A-1)p_2}{2Ap_1(1 - e^{-\frac{p_2}{Ap_1}})} \right) \quad k x_\ell^{*\alpha} = p_2 + p_1 \left(A - \frac{(2A-1)p_2}{2Ap_1(1 - e^{-\frac{p_2}{Ap_1}})} \right)$$

Continuity of the expected output distribution at x_ℓ^* gives

$$\frac{1}{p_1}(\frac{1}{a_h^\alpha} k (x_\ell^*)^\alpha + K_h(p_1,p_2) - v_h(p_2)) = A - (A - \frac{1}{p_1} k x_{h*}^\alpha) e^{-\frac{p_2}{Ap_1}}.$$

Then the constant associated with the third interval is

$$K_h(p_1.p_2) - v_h(p_2) = p_1 - \frac{p_2}{a_h^\alpha} \left(1 + \frac{2A-1}{2A} \frac{a_h^\alpha e^{-\frac{p_2}{Ap_1}} - 1}{1 - e^{-\frac{p_2}{Ap_1}}} \right).$$

Using $F(x_h^*) = 1$ the endpoint of the upper interval is characterized by

$$k {x_h^*}^\alpha = a_h^\alpha \left(p_1 - (K_h(p_1,p_2) - v_h(p_1,p_2)) \right) = p_2 \left(1 + \left(\frac{2A-1}{2A} \right) \frac{a_h^\alpha e^{-\frac{p_2}{Ap_1}} - 1}{1 - e^{-\frac{p_2}{Ap_1}}} \right).$$

Example of parameters that fall in this category: $c(e) = e^2$, $a_h = 2$ and $p_1 = 1$ and $p_2 = 1$.

Second contest

As derived in the proof of Proposition 1, for given beliefs μ_w and μ_s, the expected output distribution of the weak and strong players under this parameterization of the cost function are

$$L_s^*(x) = \begin{cases} \frac{kx^\alpha}{p_2(1-\mu_s)}, & 0 \le x \le x_s^* \\ 1, & x_s^* \le x \le x^* \end{cases} \quad H_s^*(x) = \begin{cases} 0, & 0 \le x \le x_s^* \\ \frac{kx^\alpha - k{x_s^*}^\alpha}{p_2\mu_s}, & x_s^* \le x \le x_w^* \\ 1 - \frac{k{x^*}^\alpha - kx^\alpha}{a_h^\alpha p_2 \mu_w}, & x_w^* \le x \le x^* \end{cases}$$

$$L_w^*(x) = \begin{cases} \frac{kx^\alpha}{p_2(1-\mu_w)} + \frac{\mu_s-\mu_w}{1-\mu_w}\left(\frac{a_h^\alpha - 1}{a_h^\alpha}\right), & 0 \le x \le x_s^* \\ 1 - \frac{k{x_w^*}^\alpha - kx^\alpha}{a_h^\alpha p_2(1-\mu_w)}, & x_s^* \le x \le x_w^* \\ 1, & x_w^* \le x \le x^* \end{cases} \quad H_w^*(x) = \begin{cases} 0, & 0 \le x \le x_w^* \\ 1 - \frac{k{x^*}^\alpha - kx^\alpha}{a_h^\alpha p_2 \mu_w}, & x_w^* \le x \le x^* \end{cases}$$

The expected output distributions are characterized by

$$F_s^*(x) = \begin{cases} \frac{k}{p_2}x^\alpha, & 0 \le x \le x_w^* \\ 1 - \frac{k{x^*}^\alpha - kx^\alpha}{a_h^\alpha p_2}, & x_w^* \le x \le x^* \end{cases}$$

$$F_w^*(x) = \begin{cases} \frac{k}{p_2}x^\alpha + \left(\frac{a_h^\alpha - 1}{a_h^\alpha}\right)(\mu_s - \mu_w), & 0 \le x \le x_s^* \\ 1 - \frac{k{x^*}^\alpha - kx^\alpha}{a_h^\alpha p_2}, & x_s^* \le x \le x^* \end{cases}$$

where

$$\begin{aligned} k{x_w^*}^\alpha &= p_2(1-\mu_w) \\ k{x_s^*}^\alpha &= p_2(1-\mu_s) \\ k{x^*}^\alpha &= p_2(1-\mu_w) + p_2\mu_w a_h^\alpha. \end{aligned}$$

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