Repeated Contests with Private Information

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September 23, 2019

Contests

Contests are used to motivate effort by giving a prize to the best contestant

- Firms compete to earn patents or awards
- Employees compete for bonus or promotion
- All-pay auctions

Contestants may have information about their own ability to win the contest

- R&D costs
- Worker productivity
- Value

Contestants use information to choose optimal effort

Dynamic Contests

Contests are often dynamic

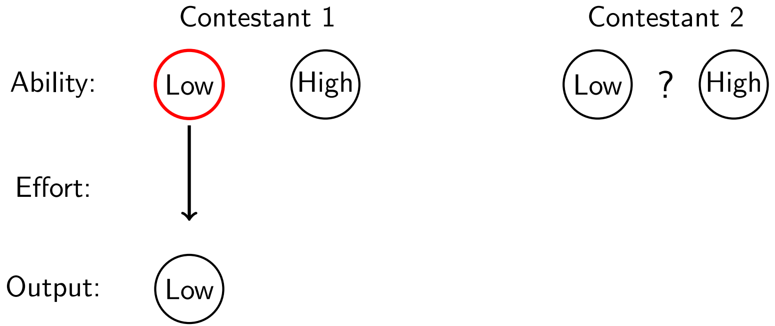
- Repeated interactions
- Occur over a period of time

Effort choices can reveal information about ability

- Information can be used later in contest
- How does this affect effort choices and outcomes of contest?
- Do contestants want to reveal ability?

Motivating Example

Single contest, opponents ability unknown.



- If Contestant 1 has low ability, chooses low effort and produces low output.

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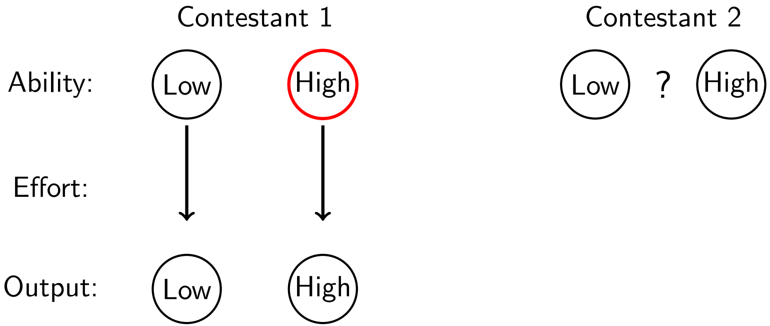

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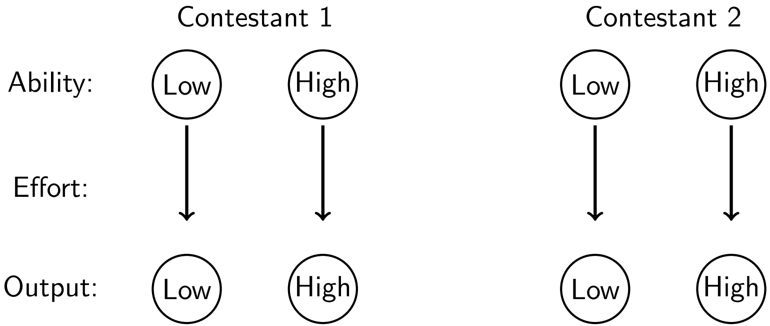
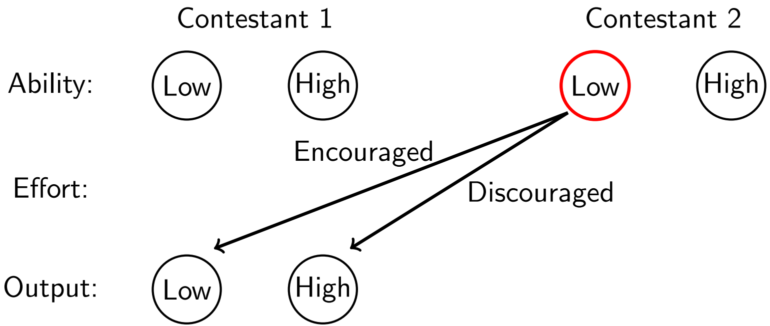


- If Contestant 1 has low ability, chooses low effort and produces low output.
- If Contestant 1 has high ability, produces high output.
- High ability contestant likely to win against low ability one.

Motivating Example

How would Contestant 2 respond to information from first contest?



- If Contestant 2 has low ability, then they are encouraged if they believe Contestant 1 also has low ability, but discouraged if they believe Contestant 1 has high ability.

Motivating Example

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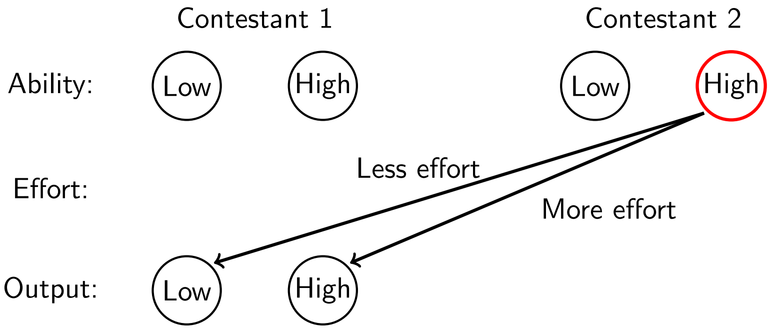
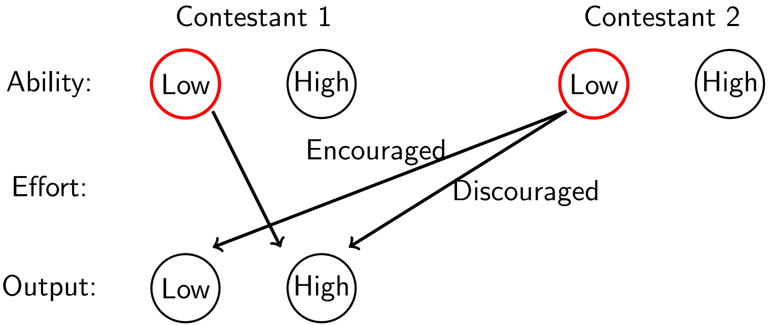


- If Contestant 2 has high ability, then they use less effort if they believe Contestant 1 also has low ability and use more effort if they believe Contestant 1 has high ability.

Motivating Example

Strategic effects of a second contest



- *Bluffing* - Contestant with low ability wants to appear strong.

Motivating Example

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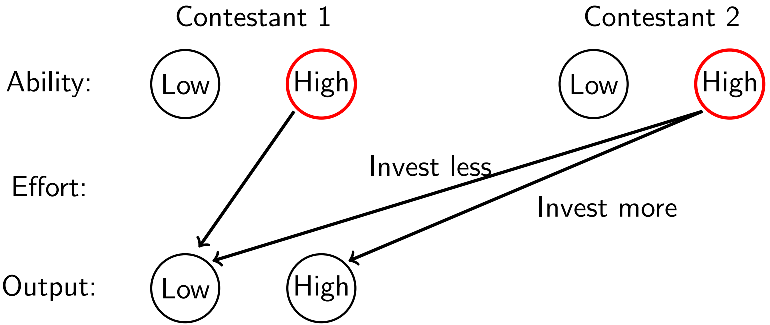

- *Sandbagging* - Contestant with high ability wants to appear weak.

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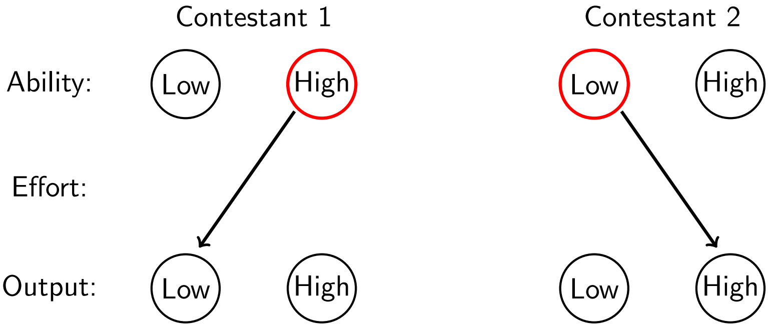

- *Bluffing* - Contestant with low ability wants to appear strong.
- *Sandbagging* - Contestant with high ability wants to appear weak.

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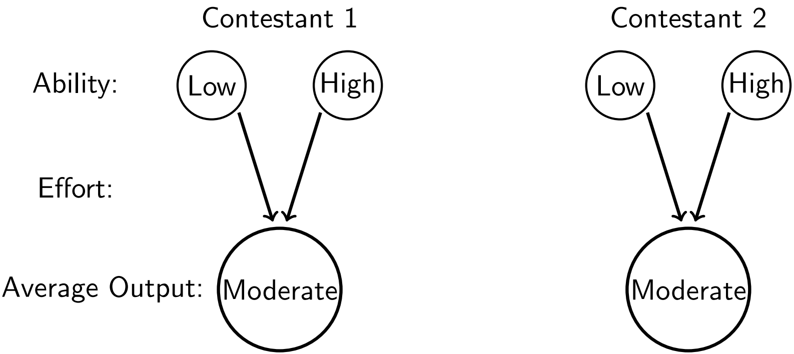
Strategic effects of a second contest



- Unique symmetric equilibrium with partial pooling: effort choice combines countervailing incentives from first and second contest.

Preview of Results

Comparison of repeated contest to single contest benchmark:

1. A high ability contestant may lose to a low ability contestant.
2. Expected aggregate output is often reduced.
   - Sandbagging and bluffing in first contest: low ability player's effort is higher and high ability player's effort is lower.
   - Increased information about ability in second contest: competitiveness is often reduced.
3. Expected payoffs of high ability contestants are higher, while low ability contestants the same.

Prize allocation to maximize output

- Prize all in one contest with linear costs (all-pay auction)
- With convex costs, uneven prizes mitigates output reduction

Selected Literature

Dynamic contests with asymmetric information:

- All-pay contests - Hörner and Sahuguet (2007), Münster (2009), Wang and Zhang (2009)
- Information disclosure - Zhang and Zhou (2016), Denter et al. (2019)

Contest design

- Dynamic competitions - Che and Gale (2003), Ederer (2010), Ridlon and Shin (2013)
- Tournaments - Rosen (1986), Fullerton and McAfee (1999), Moldovanu and Sela (2006), Ye (2007)
- First price auctions - Ortega Reichert (2001), Bergemann and Hörner (2010)

Single contest benchmark - Siegel (2014)

Model - Stage Game

- Two contestants, $i = 1, 2$
- Independently endowed with ability, $a^i \in \{a_\ell, a_h\}$
- Common prior: $Pr(a^i = a_h) = \mu_i$
- Ability is private information for each player.
- Cost of effort: $c(e^i)$, $c'(e^i) > 0$, $c''(e^i) \geq 0$.
- Output: $x(a^i, e^i) = a^i e^i$
- Payoffs in terms of output with prize p:

$$E[\pi^i(x^i, a^i)] = p\left[\Pr(x^{-i} < x^i) + \frac{1}{2}\Pr(x^{-i} = x^i)\right] - c(x^i/a^i)$$

Single Contest Benchmark

Timeline

1. Probabilities commonly known: $i = s, w$ where $\mu_s \geq \mu_w$
2. Abilities privately realized: (a^s, a^w)
3. Efforts chosen: (e^s, e^w)
4. Outputs realized: (x^s, x^w), winner determined

Equilibrium Best Response Sets

Proposition 1

There is a unique Bayesian Nash equilibrium in mixed strategies where, for each player,

1. the output distributions are continuous (except maybe at zero)
2. best response sets are disjoint, monotonic, and have no gaps.

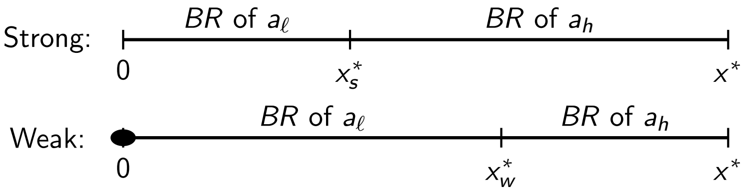


Figure 1: Best response sets of the strong and weak players.

Equilibrium Output Distributions

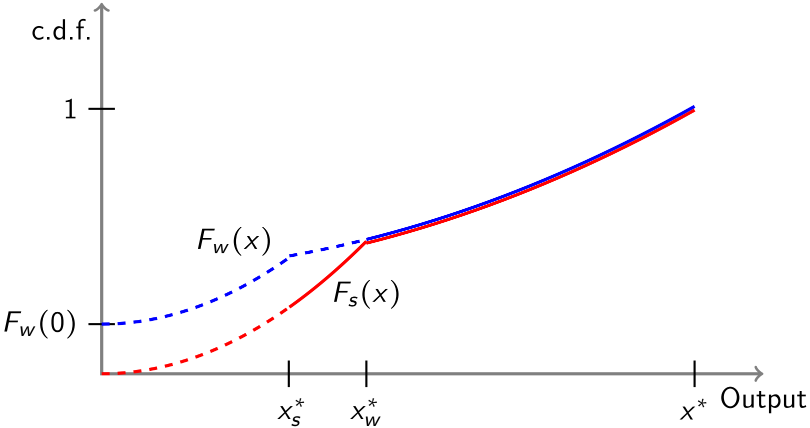


Figure 2: Expected output distributions of contestants in a single contest with quadratic costs.

Ex-interim Expected Payoffs

	High Ability	Low Ability
Strong	$p - c(x^*/a_h)$	$pF_w(0)$
Weak	$p - c(x^*/a_h)$	0

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	High Ability	Low Ability
Strong	$p - c(x^*/a_h)$	$pF_w(0)$
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- Low ability contestant expects positive payoff only when strong contestant
- High ability contestant expects same payoff as strong or weak contestant
    - Decreases with the competitiveness of the contest

Countervailing Incentives

Proposition 2

For all $\mu_i \in (0,1)$, expected payoffs in the second contest decrease for high ability players as μ_i increases, and increase with μ_i for low ability players.

	High Ability	Low Ability
Strong	$p - c(x^*/a_h)$	$pF_w(0)$
Weak	$p - c(x^*/a_h)$	0

$$p - c(x^*/a_h) = p(1-\mu_w) - c\left(\frac{c^{-1}(p(1-\mu_w))}{a_h}\right)$$

$$pF_w(0) = p(\mu_s - \mu_w) - \left[c\left(\frac{c^{-1}(p(1-\mu_w))}{a_h}\right) - c\left(\frac{c^{-1}(p(1-\mu_s))}{a_h}\right)\right]$$

Model - Repeated Contests

Timeline:

1. Common priors about ability, $\mu_1 = \mu_2 = 1/2$
2. Abilities privately realized: (a^1, a^2)
3. Stage game is played.
4. Beliefs about opponent's ability updated: $(\mu(x_1^1), \mu(x_1^2))$
5. Stage game is played.

Contestants maximize the sum of their payoffs over the two contests.

- Prizes for winning contest one, p_1, and contest two p_2, same for each contestant.
- Do not discount future payoffs.
- No direct discouragement effect.

Symmetric Perfect Bayesian Equilibrium

A set of output distributions and belief function $\{H_1^i(x_1), L_1^i(x_1), H_2^i(x_2|\mu_i,\mu_{-i}), L_2^i(x_2|\mu_i,\mu_{-i}), \mu(x)\}$ form a SPBE if

1. interim strategies are the same for each contestant,
2. contestants update beliefs about opponent's ability using Bayes' rule,
3. for every set of beliefs after first contest, strategies in the second contest are the unique Bayesian equilibrium of the single contest benchmark, and
4. given 1,2, and 3, contestants choose first period output to maximize payoffs over two contests.

Repeated Contest Equilibrium

Theorem (Uniqueness of Equilibrium)

There is a unique SPBE where in the first contest

1. *the output distributions are continuous,*
2. *the belief function is continuous and weakly increasing in output, and*
3. *best response sets for high and low ability contestants are intervals with non-trivial overlap.*

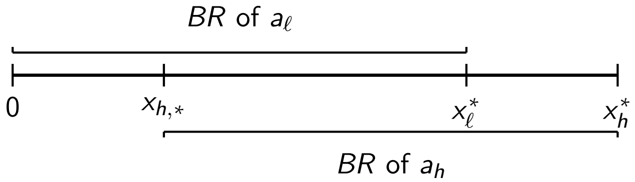
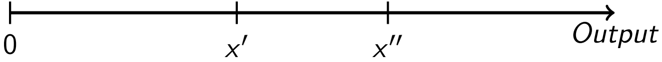


Figure 3: Best response sets in the first contest.

Monotonic Beliefs

Lemma 1

In every SPBE, $\mu(x)$ is weakly increasing in output over the support of first period strategies.



$$0 \leq \mu(x'') < \mu(x') \leq 1$$

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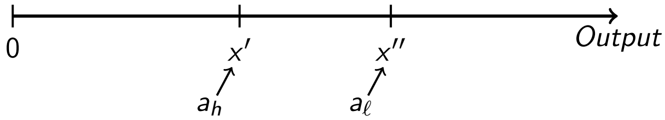

$$0 \leq \mu(x'') < \mu(x') \leq 1$$

- Current marginal costs (as in single contest)
- Value of the second contest (countervailing incentives)

Overlapping Best Response Sets

Lemma 2

The best response sets of high ability and low ability contestants are intervals whose intersection is non-empty.

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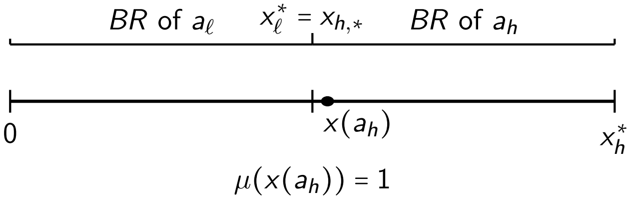

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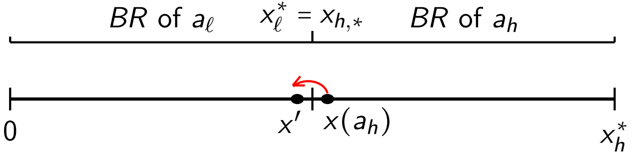


$$\mu(x(a_h)) = 1$$

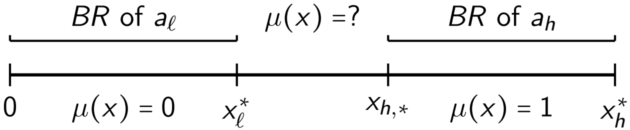
$$\mu(x') = 0$$

Payoffs continuous in first contest, discrete jump in expected payoffs in second contest

Best Response Sets are Intervals

Lemma 3

BR of a_ℓ and BR of a_h are intervals where $0 = x_{\ell,*} \leq x_{h,*} < x_\ell^* \leq x_h^*$.



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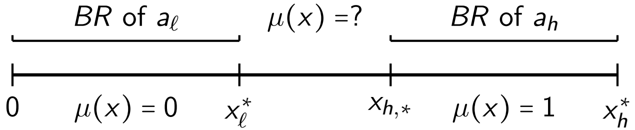

- If $\mu(x) \approx 1$, then low ability players prefer outputs off equilibrium path
- If $\mu(x) \approx 0$, then high ability players prefer outputs off equilibrium path

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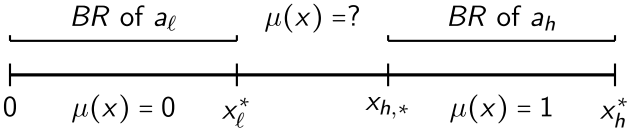

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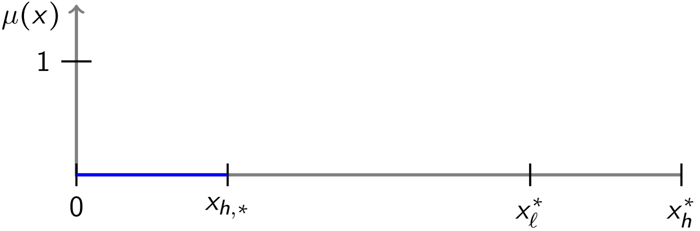
No beliefs off equilibrium path that punish both high and low ability contestants.

Equilibrium Strategies and Beliefs

Lemma 4

The belief function and the distribution functions of output, $L_1(x)$ and $H_1(x)$, are continuous in output on $[0, x_h^*]$.



- For $0 \leq x < x_{h,*}$, low ability indifferent: $F_1^*(x) = c(x)$.

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- For $x_\ell^* < x \leq x_h^*$, high ability indifferent: $F_1^*(x) = c(x_1^i / a_h) + K$.

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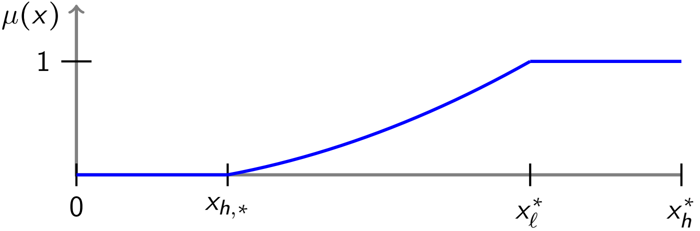


- For $0 \leq x < x_{h,*}$, low ability indifferent: $F_1^*(x) = c(x)$.
- For $x_\ell^* < x \leq x_h^*$, high ability indifferent: $F_1^*(x) = c(x_1^i/a_h) + K$.
- For $x_{h,*} \leq x \leq x_\ell^*$, low and high ability contestants indifferent. Belief function increases to offset differences in marginal cost.

Equilibrium Strategies and Beliefs

Belief function on $[x_{h,*}, x_\ell^*]$ must satisfy

$$\mu'(x)d(\mu(x)) = c'(x) - \frac{1}{a_h}c'\left(\frac{x}{a_h}\right),$$

where $d(\mu(x))$ is difference in marginal benefit of increase in the belief function:

$$d(\mu_i) \equiv \frac{\partial}{\partial \mu_i}E[v_i(\mu_i, \mu_{-i}, a_\ell)] - \frac{\partial}{\partial \mu_i}E[v_i(\mu_i, \mu_{-i}, a_h)]$$
$$= \left[p_2 + \frac{\partial}{\partial \mu_i}c\left(\frac{c^{-1}(p_2(1-\mu_i))}{a_h}\right)\right].$$

Expected output distribution on $[x_{h,*}, x_\ell^*]$ must satisfy

$$p_1 f_1^*(x) = c'(x)(1 - F_1^*(x)) + \frac{1}{a_h}c'\left(\frac{x}{a_h}\right)F_1^*(x).$$

Equilibrium Construction

The unique equilibrium can be constructed when the cost function is $c(x) = kx^{\alpha}$, with $\alpha \geq 1$ and $k > 0$.

Output distribution on $[x_{h,*}, x_{\ell}^{*}]$

$$F_1^*(x) = \frac{a_h^{\alpha}}{a_h^{\alpha} - 1} - \left(\frac{a_h^{\alpha}}{a_h^{\alpha} - 1} - \frac{1}{p_1} k x_{h,*}^{\alpha} \right) e^{-\frac{(a_h^{\alpha} - 1)}{a_h^{\alpha}} \frac{p_2}{p_1} (\mu(x) - \mu(x_{h,*}))},$$

with belief function $\mu(x) = \frac{1}{p_2} c(x) + \mu(x_{h,*})$.

Prize Structure and Information

Proposition

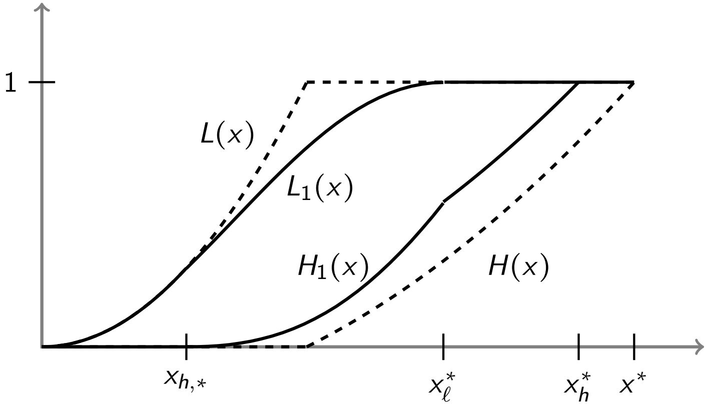
Increasing the prize of the second contest leads to a belief distribution for the second contest that second-order stocastically dominates.

Increasing the prize of the first contest leads to a belief distribution that is second order stocastically dominated.

The expected belief distribution prior to second contest:

$$Pr(\mu \leq m) = F_\mu(m) = \begin{cases} 0, & m < 0 \\ F_1(\mu{-}1(x)), & 0 \leq m < 1 \\ 1, & m \geq 1 \end{cases} .$$

Bluffing and Sandbagging



Distribution of strategies in the first of two contests with no second contest price (dashed) and same prize in each contest.

Surprise Victories

Corollary

A low ability player has a higher likelihood of winning the first contest when stakes in first contest are lower or when stakes of the second contest are higher.

- In single contest with symmetric priors, high ability contestant always wins against a low ability contestant.
- In face of future contests, high ability contestants avoid ramping up competition by not revealing type in first contest

Payoffs

Proposition

The expected payoff for a low ability contestant is zero for any set of prizes.

A high ability contestant has a higher expected payoff with an evenly split prize purse than when the entire prize is allocated to only one contest.

Low ability contestant

- No effort (zero output) always in BR set in first contest
- Zero output leads to min belief and no zero expected payoffs in second contest

High ability contestant

- Sandbagging and bluffing are complementary: attain a more favorable position in second contest at lower cost

Expected Output

Simulation of benchmark single contest vs. first and second of two repeated contests.

In the first contest, lowered output of high ability player offsets the increased output of the low ability player

- When abilities are sufficiently differentiated, reduction of output from sandbagging dominates

In the second contest, increased information on average will reduce competition

- For fixed belief about weaker contestant, higher beliefs about stronger contestant lowers expected aggregate effort.

Results - Reduced Output

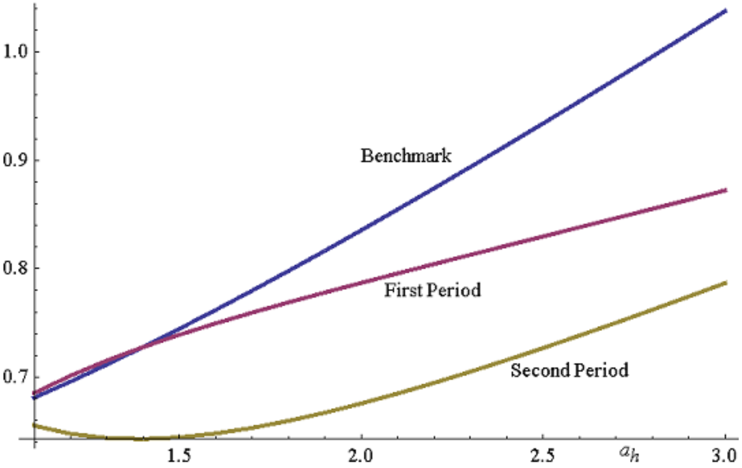


Figure: Output in terms of ability ratio, cost: $c(e) = e^2$.

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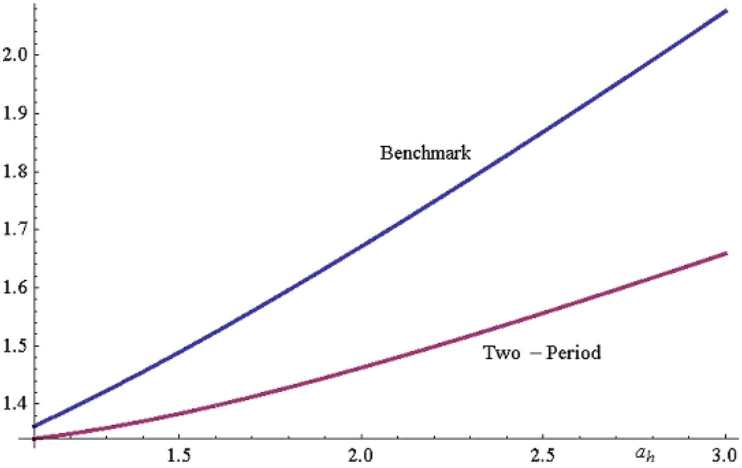

Figure: Output in terms of ability ratio, cost: $c(e) = e^2$.

Results - Increased Payoffs

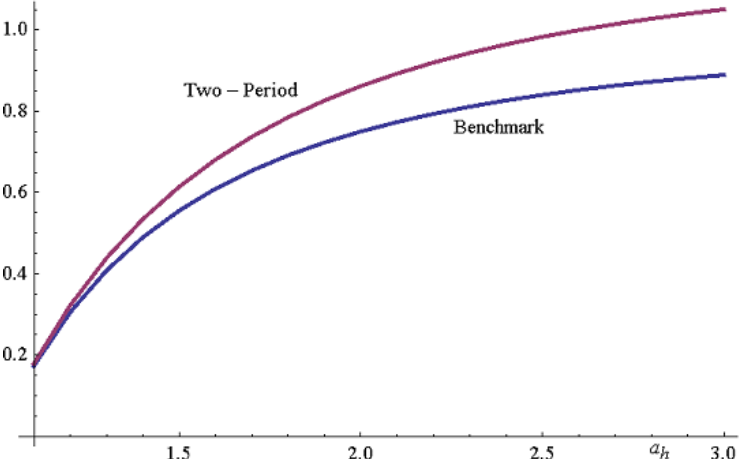

Figure: Payoffs of high ability player in terms of ability ratio, cost: $c(e) = e^2$.

Prize Allocation

Simulations with allocation of a fixed prize purse over the two contests

- Sum of prizes over two contests normalized to one

Prizes split over two contests increase expected payoffs and can reduce expected output

- Convex costs lower output when most stakes are placed on one contest

Payoffs

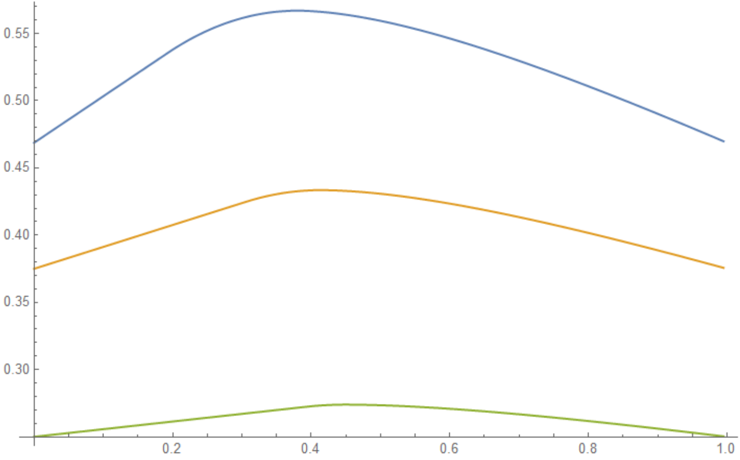

Figure: Expected payoffs against percentage of total prize allocated to first contest: $c(e) = ke$ (green), $c(e) = ke^2$ (yellow) and $c(e) = ke^4$ (blue).

Output - All Pay Auction

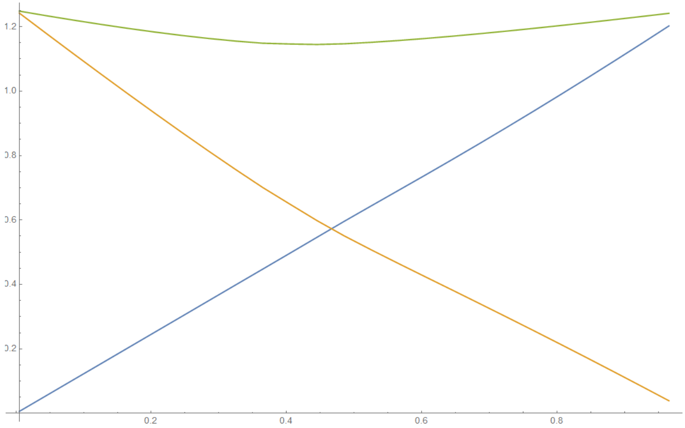

Figure: Expected total output (green), second contest output (yellow) and first contest output (blue) against percentage of total prize allocated to first contest.

Output - Convex cost

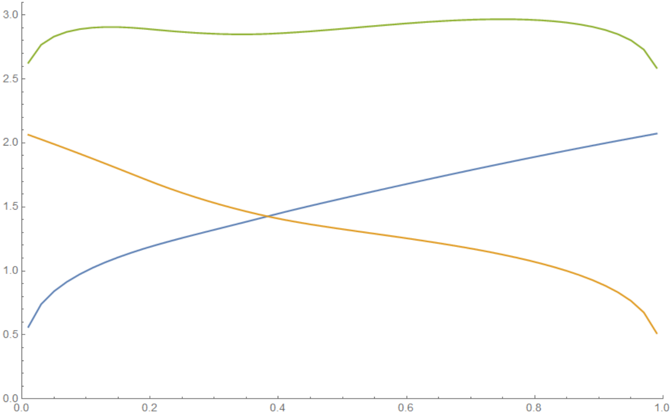

Figure: Expected total output (green), second contest output (yellow) and first contest output (blue) against percentage of total prize allocated to first contest with $c(x) = x^4$.

Conclusion

In unique equilibrium of repeated contests sandbagging and bluffing are the result of countervailing incentives from future competition

- Both high and low ability players benefit from hiding information about their ability.
- Low ability players may beat high ability players.
- Higher payoffs for players with higher ability and reduced expected output for some parameters and prizes

Prize allocation to maximize output

- Prize all in one contest with linear costs (all-pay auction)
- With convex costs, uneven prizes mitigates output reduction from repeated contests

The End

Thanks!

Repeated Contests

Definition (Symmetric Perfect Bayesian Equilibrium)

A set of output distributions $\{H_1^i(x_1), L_1^i(x_1), H_2^i(x_2|\mu_i,\mu_{-i}), L_2^i(x_2|\mu_i,\mu_{-i})$ for $i = 1, 2\}$ form a SPBE if

1. $H_1^1(x) = H_1^2(x)$, $L_1^1(x) = L_1^2(x)$,
2. $\mu_i = \mu(x_1^i) = \frac{h_1(x_1^i)}{h_1(x_1^i)+\ell_1(x_1^i)}$, for $i = 1, 2$,
3. for every (μ_i, μ_{-i}),
$$(H_2^i(x|\mu_i,\mu_{-i}), L_2^i(x|\mu_i,\mu_{-i})) = \begin{cases} (H_w^*(x|\mu_i,\mu_{-i}), L_w^*(x|\mu_i,\mu_{-i})), & \text{if } \mu_i \le \mu_{-i} \\ (H_s^*(x|\mu_i,\mu_{-i}), L_s^*(x|\mu_i,\mu_{-i})), & \text{if } \mu_i > \mu_{-i} \end{cases}$$
4. for every $x_1^i \in X_1^{\theta,i}$, $\theta = \ell, h$, player i chooses an
$$x_1^i \in \arg\max_{x^i} E[\pi(x_1^i, x_2^i(\mu(x_1^i), \mu(x_1^{-i}), a_\theta), a_\theta)] \equiv BR_i(a_\theta).$$

Countervailing Incentives

Proof: Given (μ_i, μ_j), expected payoffs for each player of each ability are given by

$$v_i(\mu_i, \mu_j, 1) = \begin{cases} \mu_i - \mu_j - \left[c\left(\frac{c^{-1}(1-\mu_j)}{a_h} \right) - c\left(\frac{c^{-1}(1-\mu_i)}{a_h} \right) \right], & \mu_i \geq \mu_j \\ 0, & \text{otherwise} \end{cases}$$

$$v_i(\mu_i, \mu_j, a_h) = 1 - \min\{\mu_i, \mu_j\} - c\left(\frac{c^{-1}(1 - \min\{\mu_i, \mu_j\})}{a_h} \right).$$

Countervailing Incentives

Proof: Given (μ_i, μ_j), expected payoffs for each player of each ability are given by

$$v_i(\mu_i, \mu_j, 1) = \begin{cases} \mu_i - \mu_j - \left[c\left(\frac{c^{-1}(1-\mu_j)}{a_h}\right) - c\left(\frac{c^{-1}(1-\mu_i)}{a_h}\right)\right], & \mu_i \geq \mu_j \\ 0, & \text{otherwise} \end{cases}$$

$$v_i(\mu_i, \mu_j, a_h) = 1 - \min\{\mu_i, \mu_j\} - c\left(\frac{c^{-1}(1 - \min\{\mu_i, \mu_j\})}{a_h}\right).$$

$$\frac{\partial}{\partial \mu_i} E[v_i(\mu_i, \mu_j, a_h)|\mu_i] = d(\mu_i)(F_{\mu_j}(\mu_i) - 1) < 0$$

$$\frac{\partial}{\partial \mu_i} E[v_i(\mu_i, \mu_j, 1)|\mu_i] = d(\mu_i)F_{\mu_j}(\mu_i) > 0$$

where $d(\mu_i) \equiv \left[1 + \frac{\partial}{\partial \mu_i} c\left(\frac{c^{-1}(1-\mu_i)}{a_h}\right)\right]$ and $F_{\mu_{-i}}(\mu_i) = \Pr(\mu_{-i} \leq \mu_i)$.

$c'(e) > 0$ and $c''(e) \geq 0$, imply $d(\mu_i) \in \left[\frac{a_h - 1}{a_h}, 1\right)$. Back